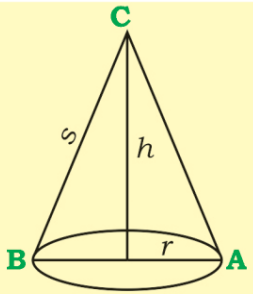
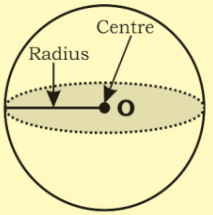
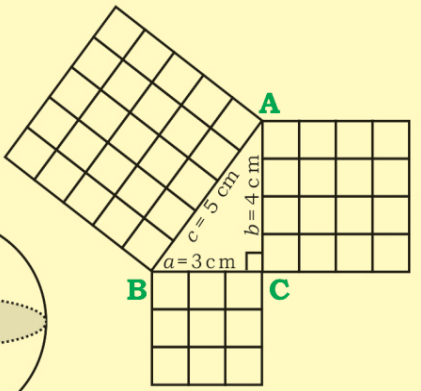
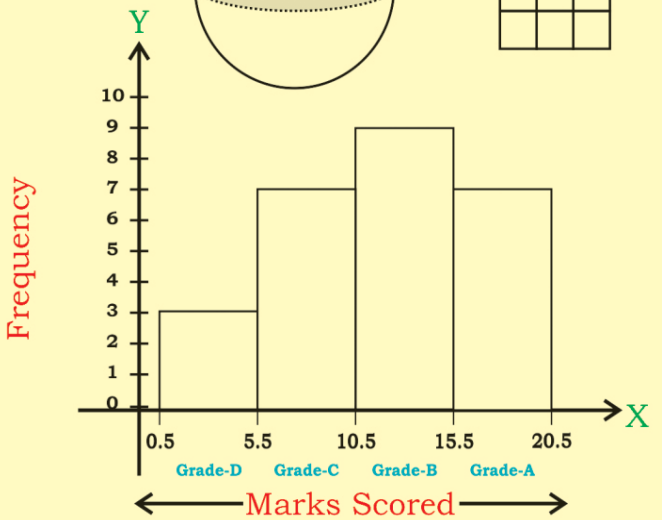
Mathematics 8

For Class VIII









Sindh Textbook Board, Jamshoro

Published by:

Prepared and produced by
The Sindh Textbook Board, Jamshoro, Sindh.

Approved by
The Bureau of Curriculum and Extension Wing Sindh, Jamshoro
Education and Literacy Department, Govt. of Sindh,
under Notification No. SO(G-I) E&L/CURRICULUM-2014 dated 8th Dec 2016,
as a sole Textbook for English Medium Schools in the Province of Sindh

Reviewed by
The Provincial Committee for Review of Textbooks
Bureau of Curriculum and Extension Wing Sindh, Jamshoro

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Year of Edition: 2024

Composing & Layout Designing
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- Farhan Ali Bhatti (Hashir Printer's and Publishers Hyd).

Printed at: **Sindh Offset Printers & Publishers**

PREFACE

The Sindh Textbook Board is an organization charged with the preparation and publication of textbooks in the province of Sindh. Its prime objective is to develop and produce textbooks which are conductive to equip the new generation with the knowledge and acumen to prepare them to face the challenges of the rapidly changing environment. In this age of knowledge explosion and development of technology not witnessed in the human history, efforts have to be made to ensure that our children do not lag behind. The Board also strives to ensure that Universal Islamic Ideology, culture and traditions are not compromised in developing the textbooks.

To accomplish this noble task, a team of educationists, experts, working teachers and friends endeavor tirelessly to develop text and improve contents, layout and design of the textbooks.

An attempt has made in this textbook to provide horizontal and vertical integration. The efforts of our experts and production personnel can bring about the desired results only if these textbooks are used effectively by teachers and students. Their suggestions will help us in further improving the qualitative contents of textbooks.

Chairman
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CONTENTS

UNIT 1

OPERATIONS ON SETS

1.1 SETS

Set is a well-defined collection of distinct objects. Sets are usually denoted by capital letters of English alphabet.Some examples are:

(1) A = Set of first six prime numbers (Descriptive form)

(2) B = {1, 2, 3, 4, 5} (Tabular form)

(3) $C = \{x \mid x \in \mathbb{Z} \wedge -5 < x < 5\}$ (Set-builder form)

REVISION EXERCISE

1. Write the following sets in tabular and set builder forms.

(i) A= Set of first five natural numbers

(ii) B= Set of integers between – 2 and 3.

(iii) C=Set of prime numbers between 10 and 30.

2. From the sets given below identify and write:

(i) Finite sets (ii) Infinite sets

A={1,2,3,4,5}, B={1,3,5,7,...}

C= Set of composite numbers between 10 and 50.

D= Set of all multiples of 10 and $E = \{x \mid x \in \mathbb{Z} \wedge x < 0\}$

3. Define with examples:

(i) Disjoint sets (ii) Overlapping sets (iii) Complement of a set

4. Verify with examples: (i) $A \cap A' = \varnothing$ (ii) $A \cup A' = \mathbf{U}$ (iii) $(A')' = A$

1.1.1 Recognize Some Important Sets:

Some important sets of numbers along with their notations are given as under:

- Set of Natural Numbers $\mathbb{N} = \{1,2,3,...\}$
- Set of Whole Numbers $\mathbb{W} = \{0,1,2,3,...\}$
- Set of Integers $\mathbb{Z} = \{...-3,-2,-1,0,1,2,3,...\}$ or $Z = \{0, \pm1, \pm2, \pm3, ...\}$
- Set of Even Integers $\mathbb{E} = \{...,-6,-4,-2,0,2,4,6,...\}$ or $E = \{0, \pm2, \pm4, \pm6, ...\}$
- Set of Odd Integers $\mathbb{O} = \{...,-5,-3,-1, 1,3,5,...\}$ or $O = \{\pm1, \pm3, \pm5, ...\}$
- Set of Prime Numbers $\mathbb{P} = \{2,3,5,7,...\}$
- Set of Composite Numbers $\mathbb{C} = \{4, 6, 8, 9, 10, 12,...\}$
- Set of Rational Numbers $\mathbb{Q} = \{x \mid x = \frac{p}{q}, p \wedge q \in \mathbb{Z}, q \neq 0\}$

1.1.2 Find a subset of a set

Subset: Set A is said to be a subset of a set B if every element of A is also an element of B. Symbolically, we write as $A \subseteq B$.

Example. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$ we see that $A \subseteq B$

Notes: **(i)** Empty set is a subset of every set $\varnothing \subseteq A$.

(ii) Every set is a subset of itself i.e. $A \subseteq A$.

1.1.3 Define Proper subset and an Improper subset

1. Proper subset

Let set A is a subset of set B but B is not a subset of A, then A is called proper subset of B, denoted by $A \subset B$.

Example 1. Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$

Here, $B \subseteq A$ but $A \not\subseteq B \Rightarrow B \subset A$.

Note: If B is not a proper subset of a set A, then we write it as $B \not\subset A$.

2. Improper subset

Two sets A and B are said to be improper subsets of each other if $A \subseteq B$ and $B \subseteq A$.

Note: There is no any particular symbol for an improper subset.

Example 2. Let $A = \{1, 2, 3, 4\}$ and $B = \{4, 3, 2, 1\}$.

Here A and B are improper subsets of each other.

Example 3. Find all possible subsets of $A = \{x, y, z\}$

All possible subsets are:

$$\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}$$

Here Number of subsets $= 2^3 = 8$.

1.1.4. Find power set P(A) of a set A.

Power Set: Let A be a set. The set of all possible subsets of A is called power set of A. It is denoted by P (A).

Example 4. Find the power set of $A = \{a, b\}$.

Solution: Here, number of subsets $= 2^2 = 4$. All possible subsets are: $\varnothing, \{a\}, \{b\}, \{a, b\}$. So, $P(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

Note: (i) Number of all possible subsets of an empty set is $1 = 2^0$

(ii) Number of all possible subsets of a set having one element is $2 = 2^1$

(iii) Number of all possible subsets of a set having two elements is $4 = 2^2$

(iv) Number of all possible subsets of a set having three elements is $8 = 2^3$

$\therefore$ Numbers of all possible subsets of a set having "n" elements is 2^n

EXERCISE 1.1

1. Indicate which one is a subset of the other

(i) $A = \{5,6,7\}$ and $B = \{1,2,3,...,10\}$

(ii) $C = \{2\}$ and $D = \{2,3,5\}$

(iii) $T = \{1,2,3,..., 8\}$ and $S = \{2,4,6\}$

2. Find any three proper subsets of:

(i) $A = \{a, e, i, o, u\}$ (ii) $B = \{x, y\}$

3. Find any two proper subsets and one improper subset of:

(i) $X = \{2, 4\}$ (ii) Y = Set of prime numbers less than 4

4. Find all possible subsets of:

(i) $F = \{1,2,3\}$ (ii) A = Set of even prime numbers

(iii) B = Set of odd prime numbers less than 7

(iv) $C = \{x / x \in \mathbb{Z} \wedge -2 < x < 2\}$ (v) $D = \{x / x \in \mathbb{E}^{+} \wedge x \leqslant 6\}$

5. Find the power set of:

(i) $A = \{1, 3, 5\}$ (ii) $B = \{a, b, c, d\}$ (iii) $C = \varnothing$

6. Find a set which has only one proper subset.

7. Find a set which has no proper subset at all.

8. Find improper subsets of the following:

(i) $A = \{x / x \in \mathbb{O} \wedge x \leqslant 7\}$ (ii) $B = \{x / x \in \mathbb{P} \wedge x \leqslant 11\}$

9. Let X = {20, 30, 40, 50, 60, 80, 100}. Find the number of elements of the following subsets of X

(i) A = Set of numbers divisible by 2.

(ii) B = Set of numbers divisible by 8.

(iii) C = Set of numbers divisible by 15.

(iv) D = Set of numbers divisible by 25.

10. Decide which one is a subset of the other.

(i) $\mathbb{N}$ and $\mathbb{W}$ (ii) $\mathbb{Z}$ and $\mathbb{N}$ (iii) $\mathbb{P}$ and $\mathbb{N}$

(iv) $\mathbb{E}$ and $\mathbb{Z}$ (v) $\mathbb{Q}$ and $\mathbb{O}$ (vi) $\mathbb{Q}$ and $\mathbb{E}$

1.2 OPERATIONS ON SETS

1.2.1 Verify commutative and associative laws with respect to union and intersection.

(A) Commutative laws w.r.t union and intersection.

Let A and B be any two sets.

Then commutative law w.r.t. union is: $A \cup B = B \cup A$

And commutative law w.r.t. intersection is: $A \cap B = B \cap A$

Example 1. If A = {1, 2, 3} and B = {1, 3, 5}, then verify commutative laws w.r.t. union and intersection.

Solution: (i) Commutative law w.r.t union

We have to show that $A \cup B = B \cup A$

L.H.S. $= A \cup B$
$= \{1, 2, 3\} \cup \{1, 3, 5\}$
$= \{1, 2, 3, 5\}$

R. H.S.$= B \cup A$
$= \{1, 3, 5\} \cup \{1, 2, 3\}$
$= \{1, 2, 3, 5\}$

As L.H.S. = R.H.S.

$\therefore A \cup B = B \cup A.$ Hence verified

Solution: (ii) Commutative law w.r.t. intersection

We have to show that $A \cap B = B \cap A$

L.H.S. $= A \cap B$
$= \{1, 2, 3\} \cap \{1, 3, 5\}$
$= \{1, 3\}$

R. H.S. $= B \cap A$
$= \{1, 3, 5\} \cap \{1, 2, 3\}$
$= \{1, 3\}$

As L.H.S. = R.H.S.

$\therefore A \cap B = B \cap A.$ Hence verified

(B) Associative laws w.r.t. union and intersection.

Let A, B and C be any three sets,

then associative law w.r.t. union is: $A \cup (B \cup C) = (A \cup B) \cup C$

And associative law w.r.t. intersection is: $A \cap (B \cap C) = (A \cap B) \cap C$

Example 2. If A = {1, 2, 3}, B = {2, 3, 4} and C = {3, 4, 5}, then verify:

(i) $A \cap (B \cap C) = (A \cap B) \cap C$ (ii) $A \cup (B \cup C) = (A \cup B) \cup C$.

Solution:(i) $A \cap (B \cap C) = (A \cap B) \cap C$. [Associative law w.r.t. intersection]

L.H.S.	R.H.S.
$= A \cap (B \cap C)$	$= (A \cap B) \cap C$
$= \{1, 2, 3\} \cap (\{2, 3, 4\} \cap \{3, 4, 5\})$	$= (\{1, 2, 3\} \cap \{2, 3, 4\}) \cap \{3, 4, 5\}$
$= \{1, 2, 3\} \cap \{3, 4\} = \{3\}$	$= \{2, 3\} \cap \{3, 4, 5\} = \{3\}$

As L.H.S. = R.H.S.

$\therefore$ $A \cap (B \cap C) = (A \cap B) \cap C$. Hence verified.

Solution: (ii) $A \cup (B \cup C) = (A \cup B) \cup C$. [Associative Law w.r.t. to union]

L.H.S.	R.H.S.
$= A \cup (B \cup C)$	$= (A \cup B) \cup C$
$= \{1, 2, 3\} \cup (\{2, 3, 4\} \cup \{3, 4, 5\})$	$= (\{1, 2, 3\} \cup \{2, 3, 4\}) \cup \{3, 4, 5\}$
$= \{1, 2, 3\} \cup \{2, 3, 4, 5\}$	$= \{1, 2, 3, 4\} \cup \{3, 4, 5\}$
$= \{1, 2, 3, 4\} \cup \{3, 4, 5\}$	$= \{1, 2, 3, 4, 5\}$
$= \{1, 2, 3, 4, 5\}$	

As L.H.S. = R.H.S. Hence verified.

$\therefore$ $A \cup (B \cup C) = (A \cup B) \cup C$.

1.2.2 To verify distributive laws

Let A, B and C be any three sets. Then distributive law of union over intersection is:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

And distributive law of intersection over union is:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example:

If $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$ and $C = \{2, 3, 6\}$, then verify distributive laws.

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S. $= A \cup (B \cap C) = \{1, 2, 3\} \cup (\{2, 4, 6\} \cap \{2, 3, 6\})$

$= \{1, 2, 3\} \cup \{2, 6\} = \{1, 2, 3, 6\}$.

R.H.S. $= (A \cup B) \cap (A \cup C) = (\{1, 2, 3\} \cup \{2, 4, 6\}) \cap (\{1, 2, 3\} \cup \{2, 3, 6\})$

$= \{1, 2, 3, 4, 6\} \cap \{1, 2, 3, 6\} = \{1, 2, 3, 6\}$.

As L.H.S. = R.H.S.

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Hence verified.

Solution: (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

L.H.S. $= A \cap (B \cup C) = \{1, 2, 3\} \cap (\{2, 4, 6\} \cup \{2, 3, 6\})$

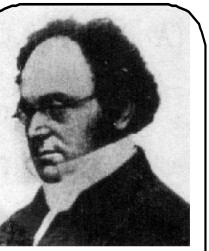
$= \{1, 2, 3\} \cap \{2, 3, 4, 6\} = \{2, 3\}$.

R.H.S. $= (A \cap B) \cup (A \cap C) = (\{1, 2, 3\} \cap \{2, 4, 6\}) \cup (\{1, 2, 3\} \cap \{2, 3, 6\})$

$= \{2\} \cup \{2, 3\} = \{2, 3\}$.

L.H.S. = R.H.S.

$\therefore$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Hence verified.



August De Morgan (1806-1871) a British mathematic who formulated De Morgan Law.

1.2.3 State and Verify De Morgan's Laws.

Let A and B be subsets of a universal set **U**.

Then De Morgan's Laws are:

(i) $(A \cup B)' = A' \cap B'$, (ii) $(A \cap B)' = A' \cup B'$.

Example. If **U** $= \{1, 2, 3, \ldots, 6\}$, $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$, verify any one of the De Morgan's Laws.

Solution: One of the De Morgan's Law is: $(A \cap B)' = A' \cup B'$

L.H.S. Here $A \cap B = \{1, 3, 5\} \cap \{2, 4, 6\} = \{\}$

Therefore $(A \cap B)' = \mathbf{U} - (A \cap B) = \{1, 2, 3, \ldots, 6\} - \{\}$

$= \{1, 2, 3, \ldots, 6\}$ = L.H.S.

R.H.S. Here $A' = \mathbf{U} - A$

$= \{1, 2, 3, \ldots, 6\} - \{1, 3, 5\} = \{2, 4, 6\}$

$B' = \mathbf{U} - B = \{1, 2, 3, \ldots, 6\} - \{2, 4, 6\} = \{1, 3, 5\}$

Therefore $A' \cup B' = \{2, 4, 6\} \cup \{1, 3, 5\}$

$= \{1, 2, 3, 4, 5, 6\}$ = R.H.S.

$\therefore$ $(A \cap B)' = A' \cup B'$. Hence verified.

Activity: Let **U** $= \{a, b, c, d, e, f\}$, $A = \{a, b, c, d\}$ and $B = \{c, d, e, f\}$

Verify **(i)** $(A \cup B)' = A' \cap B'$

Here $A \cup B = \ldots\ldots\ldots$

$(A \cup B)' = \ldots\ldots\ldots$

$A' = \ldots\ldots\ldots, B' = \ldots\ldots\ldots$

$A' \cap B' \ldots\ldots$

$\therefore$ L.H.S $= \ldots\ldots$

R.H.S $= \ldots\ldots$

Thus, L.H.S $= \ldots\ldots\ldots$

(ii) $(A \cap B)' = A' \cup B'$

Here $A \cap B = \ldots\ldots\ldots$

$(A \cap B)' = \ldots\ldots\ldots$

$A' = \ldots\ldots\ldots$

$B' = \ldots\ldots\ldots$

$A' \cup B' = \ldots\ldots$

$\therefore$ L.H.S $= \ldots\ldots$

R.H.S $= \ldots\ldots$

Thus, $\ldots\ldots$ = R.H.S

EXERCISE 1.2

1. If A = {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} and C = {2, 4, 6, 8}, then verify:

(i) Commutative law w.r.t. intersection and union

(ii) Associative law w.r.t union and intersection

(iii) Distributive law of intersection over union

(iv) Distributive law of union over intersection.

2. If P = {*a*, *b*, *c*, *d*, *e*}, Q = {*a*, *e*, *i*, *o*, *u*} and R = {*c*, *d*, *e*, *i*}, then verify that

(i) $P \cup Q = Q \cup P$ (ii) $P \cap Q = Q \cap P$

(iii) $P \cap (Q \cap R) = (P \cap Q) \cap R$ (iv) $P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$

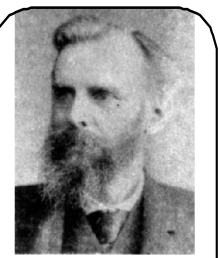
3. If U = {1, 2, 3, ..., 10}, A = {1, 2, 3, 4, 5} and B = {2, 4, 6, 8, 10}, then verify that: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

4. If $P = \{x \mid x \in \mathbb{C} \wedge x \leqslant 11\}$, $Q = \{x \mid x \in \mathbb{P} \wedge 3 \leqslant x \leqslant 11\}$ and $R = \{x \mid x \in \mathbb{E} \wedge x \leqslant 10\}$, then verify distributive laws.

5. If $U = \{x \mid x \in \mathbb{N} \wedge x \leqslant 11\}$, $A = \{x \mid x \in \mathbb{E} \wedge 2 < x < 10\}$ and $B = \{x \mid x \in \mathbb{P} \wedge 3 \leqslant x \leqslant 11\}$, then verify De-Morgan's Laws.

1.3 VENN DIAGRAM

A diagram in which we represent sets pictorially as circles within an enclosing rectangle (universal set), common elements of the sets are represented by intersection of circles, termed as Venn digram.

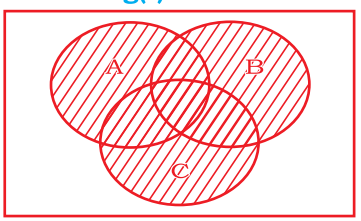


John Venn (1834-1923), an English mathematician who introduced Venn diagrams.

1.3.1 Demonstrate union and intersection of three overlapping sets through Venn Diagram

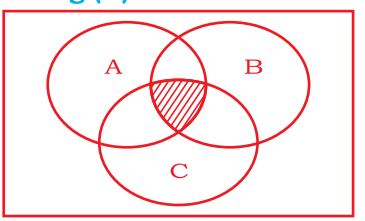
Here Union and Intersection of three overlapping sets is shown through Venn diagram in the following figures.

Fig(i): Union



Shaded portion represents Union

Fig (ii): Intersection



Shaded portion represents Intersection

1.3.2 To Verify Associative and Distributive Laws through Venn Diagram

Associative and distributive laws can easily be verified through Venn diagrams as explained in the following examples.

Example. If $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and $C = \{3, 4, 5\}$, then verify through Venn diagrams.

(i) $A \cap (B \cap C) = (A \cap B) \cap C$, (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution: (i) $A \cap (B \cap C) = (A \cap B) \cap C$

L.H.S. $= A \cap (B \cap C)$.

Here  shaded portion represents $B \cap C$.

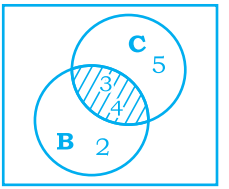


Figure (i)

According to Fig **(i)** ; $B \cap C = \{3, 4\}$

According to Fig **(ii)** ; $A \cap (B \cap C) = \{3\}$

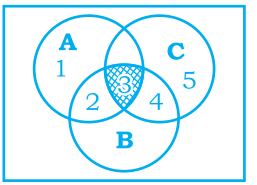


Figure (ii)

R.H.S. $= (A \cap B) \cap C$

Here  shaded portion represents $A \cap (B \cap C)$.

According to Fig **(iii)** ; $A \cap B = \{2, 3\}$

Here  shaded portion represents $(A \cap B)$.

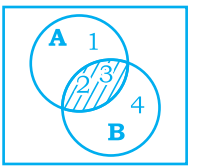


Figure (iii)

And according to Fig **(iv)** ; $(A \cap B) \cap C = \{3\}$

Thus, L.H.S. = R.H.S.

$\therefore$ $A \cap (B \cap C) = (A \cap B) \cap C$.

Here  shaded portion represents $(A \cap B) \cap C$.

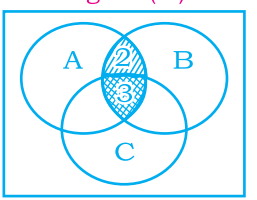


Figure (iv)

Hence verified, the associative law w.r.t. Intersection.

Activity: Verify the Associative Law w.r.t. union.

Solution: (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S. $= A \cup (B \cap C)$.

Here  shaded portion represents $B \cap C$.

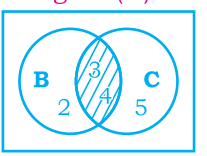


Fig (v)

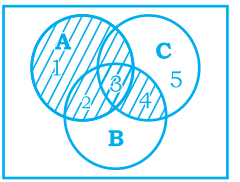
According to Fig (v); $B \cap C = \{3, 4\}$.

According to Fig (vi); $A \cup (B \cap C) = \{1, 2, 3, 4\}$

R.H.S. = $(A \cup B) \cap (A \cup C)$

Here and shaded portion represents $A \cup (B \cap C)$.

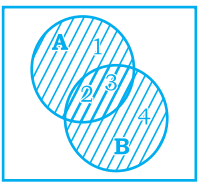
Fig (vi)



According to Fig (vii); $A \cup B = \{1, 2, 3, 4\}$

Here shaded portion represents $A \cup B$

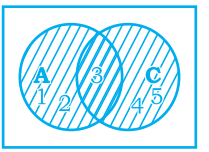
Fig (vii)



According to Fig (viii); $A \cup C = \{1, 2, 3, 4, 5\}$.

Here shaded portion represents $A \cup C$

Fig (viii)



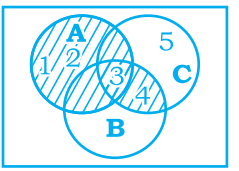
According to Figure (ix) $(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4\}$.

Thus, L.H.S. = R.H.S.

Here shaded portion represents $(A \cup B) \cap (A \cup C)$

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Hence verified, The Distributive Law of Union over Intersection.

Fig (ix)



Activity: Students themselves verify Distributive Law of Intersection over union through Venn Diagram..

EXERCISE 1.3

1. If $A = \{5, 6, 7\}$, $B = \{6, 7, 8\}$ and $C = \{7, 8, 9\}$, then represent:

(i) $A \cup (B \cup C)$ (ii) $A \cap (B \cap C)$ through Venn diagrams.

2. If A = Set of even natural numbers less than 15
B = Set of Prime numbers less than 15 and
C = Set of Odd natural numbers less than 15 then show the following by using Venn diagrams:

(i) $A \cup (B \cup C)$ (ii) $A \cap (B \cap C)$ (iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3. If $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{c, d, e\}$, verify through Venn diagram that: (i) $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (iii) $A \cap (B \cap C) = (A \cap B) \cap C$

4. If $D = \{x \mid x \in \mathbb{N} \wedge x \leqslant 10\}$, $E = \{x \mid x \in \mathbb{W} \wedge x \leqslant 8\}$

$F = \{x \mid x \in \mathbb{P} \wedge x \leqslant 11\}$, then verify by using Venn diagram.

(i) $D \cup (E \cap F) = (D \cup E) \cap (D \cup F)$ (ii) $D \cap (E \cup F) = (D \cap E) \cup (D \cap F)$

REVIEW EXERCISE 1

1. Choose the correct answer.

(i) Which one is a subset of $\mathbb{W}$?

(a) $\mathbb{Z}$ (b) $\mathbb{N}$ (c) $\mathbb{Q}$ (d) $\mathbb{E}$

(ii) Number of all possible subsets of $A = \{a, e, i, o, u\}$ is _____.

(a) 8 (b) 16 (c) 32 (d) 64

(iii) For any three sets A, B and C, $(A \cup B) \cap (A \cup C)$ = _____.

(a) $A \cap (B \cup C)$ (b) $A \cup (B \cup C)$ (c) $A \cup (B \cap C)$ (d) None of these

(iv) In Venn diagram, universal set is usually denoted by _____.

(a) Triangle (b) Rectangle (c) Circle (d) Star

2. Find P(A) if $A = \{x, y, z\}$.

3. If $A = \{1, 2, 3\}$, $B = \{2, 4, 6\}$ and $C = \{x | x \in \mathbb{W} \wedge x \leqslant 4\}$ then verify:

(i) $(A \cap B) \cap C = A \cap (B \cap C)$ (ii) $(A \cup B) \cup C = A \cup (B \cup C)$

(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (iv) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

(v) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ (vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4. If $U = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$ then verify De Morgan's Laws.

5. Choose the correct answer.

(i) $A = \{2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 9\}$, then $A \cap B$ is:

(a) $\{2, 4\}$ (b) $\{\ \}$ (c) $\{1, 3\}$ (d) $\{2, 9\}$

(ii) $A = \{0, 2\}$, $B = \{1, 3\}$, then $A \cup B$ is:

(a) $\{0, 2\}$ (b) $\{1, 3\}$ (c) $\{\ \}$ (d) $\{0, 1, 2, 3\}$

(iii) If $P = \{0, 2, 6\}$ then improper subset of P is:

(a) $\{0, 2\}$ (b) $\{0, 6\}$ (c) $\{0, 2, 6\ \}$ (d) $\{2, 6\}$

6. Which of the following is a power set and write its original set.

(i) $\varnothing$ (ii) $\{\varnothing, x\}$ (iii) $\{\varnothing, \{x\}, \{y\}\}$
(iv) $\{\varnothing, \{x\}, \{y\}, \{x, y\}\}$ (v) $\{\varnothing, \{x\}, \{y\}, \{z\}\}$ (vi) $\{\varnothing\}$

SUMMARY

- Well-defined collection of distinct objects is called a set.
- Important sets: $\mathbb{N} = \{1, 2, 3, \ldots\}$ $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$

  $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\}$ $\mathbb{Q} = \{x | x = \frac{p}{q}, p \wedge q \in \mathbb{Z}, q \neq 0\}$

  $\mathbb{E} = \{0, \pm2, \pm4, \ldots\}$, $\mathbb{O} = \{\pm1, \pm3, \pm5, \ldots\}$, $\mathbb{P} = \{2, 3, 5, 7, \ldots\}$
- If every element of set A is also an element of set B, then A is a subset of B.
- Any subset of set A is called a proper subset of A, if it is not equal to A.
- Any subset of set A is called an improper subset of A, if it is equal to A.
- The set of all possible subsets of set A is called power set of A. It is denoted as P(A).
- Empty set is a subset of every set.
- If A consists of n elements then P(A) consist of 2^n elements.
- Commutative laws: (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$
- Associative laws: (i) $A \cup (B \cup C) = (A \cup B) \cup C$
  (ii) $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive laws: (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's laws:
  (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
- A diagram in which we represent sets pictorially as circles within an enclosing rectangle (universal set), common elements of the sets are represented by intersection of circles, termed as Venn digram.

UNIT 2

REAL NUMBERS

2.1 IRRATIONAL NUMBERS

2.1.1 Define an irrational number

We use numbers in our daily life. Almost every number of this world we can think of is a real number.

For example, $0, -1, 12.59, -0.3846, \frac{3}{4}, \pi$ are the real numbers.

Real numbers are a combination of rational and irrational numbers.

The numbers $\frac{2}{3}, \frac{1}{2}, \frac{5}{6}$ and $\frac{22}{7}$ represent the rational numbers. These numbers can be represented in the form of $\frac{p}{q}$ where p and $q \in \mathbb{Z}$ and $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$.

Irrational numbers cannot be expressed in the form of a ratio of integers as $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. If we take square root of a number which is not a perfect square, it will be called an irrational number. $\sqrt{2}, \sqrt{3}, \sqrt{8}, \pi$ are examples of irrational numbers. The Set of irrational numbers is denoted by $\mathbb{Q}'$.

Irrational numbers can be expressed in the form of decimal numbers which are non-terminating and non-recurring.

Irrational numbers are non-terminating and non-repeating decimals.

2.1.2 Recognize rational and irrational numbers

Let us recognize rational and irrational numbers with the help of examples: The numbers $\frac{2}{8}, 4.755, 5, 0, \sqrt{4}, 2\sqrt{9}, 1.9, 0.333\ldots,$ $-\sqrt{49}$ and $3\frac{2}{3}$ are rational numbers.

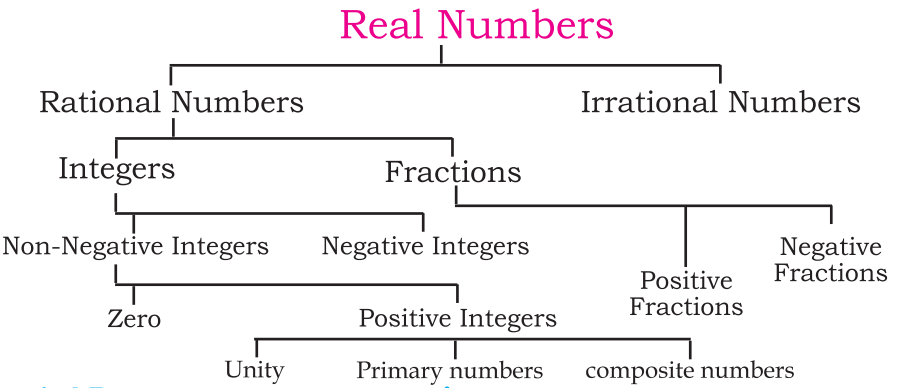
The numbers $\sqrt{\frac{2}{5}}, -3\sqrt{8}, \pi, \sqrt{7}, \sqrt{48}, \sqrt{122}$, are all irrational numbers.

$\sqrt{\frac{2}{5}}$ and $\frac{1}{\sqrt{3}}$ are irrational because in both, their numerator or denominator are not all integers.

2.1.3 Define real numbers

A real number is a value that shows any quantity mapped on a number line. Union of the sets of rational numbers ($\mathbb{Q}$) and irrational numbers ($\mathbb{Q}'$) is called the set of real numbers, denoted by $\mathbb{R}$, i.e $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

We can conclude it by this diagram.



2.1.4 Demonstrate non-terminating/non-repeating (or non - periodic) decimals

Let us consider the following examples:

Example 1. Convert $\frac{1}{3}$ into decimal.

Solution:

$$
\begin{array}{r|l}
3 & 0.333\ldots \\
\hline
 & 10 \\
 & -9 \\
 & \hline
 & 10 \\
 & -9 \\
 & \hline
 & 10 \\
 & -9 \\
 & \hline
 & 1
\end{array}
$$

So, $\frac{1}{3} = 0.333\ldots$ or $0.\overline{3}$ is a non-terminating and repeating but periodic decimal (as 3 is repeating infinitely).

Note: If remainder $\neq 0$, the number is called non-terminating decimal.

Example 2. Convert $-\frac{4}{7}$ into decimal.

Solution:

$$
\begin{array}{r|l}
7 & -0.571428 \\
\hline
 & -40 \\
 & +35 \\
 & \hline
 & -50 \\
 & +49 \\
 & \hline
 & -10 \\
 & +7 \\
 & \hline
 & -30 \\
 & +28 \\
 & \hline
 & -20 \\
 & +14 \\
 & \hline
 & -60 \\
 & +54 \\
 & \hline
 & -4
\end{array}
$$

So, $-\frac{4}{7} = -0.571428\ldots$

Example 3. Convert $\frac{1}{2}$ and $-\frac{4}{5}$ into decimal.

Solution:

(i) $\frac{1}{2} = 0.5$, it is a terminating decimal.

(ii) $\frac{-4}{5} = -0.8$, it is also a terminating decimal.

> ***Note:*** *If remainder = 0 after division then common fractions is called terminating decimal fraction.*

Hence every common fraction is either terminating or non-terminating but a repeating decimal. So every common fraction is a rational number.

Irrationals are numbers, whose digits in the decimal part are non-terminating and non-repeating in the same order, like $\sqrt{2} = 1.1412135...$, $\sqrt{5} = 2.23606...$, $\sqrt{243} = 15.5884...$ and $\pi = 3.1417...$, are called irrational numbers.

EXERCISE 2.1

A. Separate rational and irrational numbers from the following:

1. 35 **2.** $\sqrt{36}$ **3.** $-2\frac{1}{40}$ **4.** $2\sqrt{4}$ **5.** 6π **6.** $\frac{22}{7}$

7. $\frac{2}{\sqrt{5}}$ **8.** $-\sqrt{1}$ **9.** $\frac{\pi}{5}$ **10.** $\frac{\sqrt{2}}{\pi}$ **11.** $\frac{\sqrt{9}}{\sqrt{25}}$ **12.** $\sqrt{11}$

B. Identify terminating and non-terminating (periodic) decimal fractions from the following rational numbers.

1. $\frac{1}{7}$ **2.** $\frac{2}{3}$ **3.** 0 **4.** $\frac{1}{11}$ **5.** $\frac{4}{3}$ **6.** $\frac{-1}{5}$

7. $\frac{-5}{6}$ **8.** 2256 **9.** $\frac{9}{10}$ **10.** $\frac{25}{26}$ **11.** $\frac{243}{100}$ **12.** $\frac{18750}{1000}$

2.2 SQUARES

The product of a number multiplied by itself is called its square. For example square of 2 is $2^2 = 2 \times 2 = 4$. In the same way square of a is $a^2 = a \times a$.

2.2.1 Find perfect square of a number

A natural number which is square of another natural number is called a perfect square.

For example $9 = 3^2$. So, 9 is a perfect square of 3.

Similarly, $16 = 4^2$. So, 16 is a perfect square of 4.

Example 1. Find the perfect squares of 11 and 65.

Solution: Perfect square of $11 = 11^2 = 11 \times 11 = 121$

Perfect square of $65 = 65^2 = 65 \times 65 = 4225$.

Note: We can find perfect squares of rational numbers but we cannot find perfect squares of irrational numbers.

For example. $(\sqrt{2})^2 = 2$ is a square but not a perfect square.

$(\pi)^2 = \pi^2$ is not a perfect square.

2.2.2 Establish patterns for the squares of natural numbers

Let us establish a pattern of adding natural numbers to get squares of natural numbers.

The first natural number is 1, therefore $1^2 = 1 \times 1 = 1$

Now $2^2 = 2 \times 2 = 4 = 2 + 1 + 1$ or $2^2 = 1 + 2 + 1$ (by adding current number to be squared with twice the number before it).

In the same way: $3^2 = 3 \times 3 = 9 = 3 + 2 + 2 + 1 + 1$ or $3^2 = 1 + 2 + 3 + 2 + 1$

Hence the desired pattern of squares by addition is as under:

$1^2 = \mathbf{1} = 1$

$2^2 = 1 + \mathbf{2} + 1 = 4$

$3^2 = 1 + 2 + \mathbf{3} + 2 + 1 = 9$

$4^2 = 1 + 2 + 3 + \mathbf{4} + 3 + 2 + 1 = 16$

$5^2 = 1 + 2 + 3 + 4 + \mathbf{5} + 4 + 3 + 2 + 1 = 25$

$6^2 = 1 + 2 + 3 + 4 + 5 + \mathbf{6} + 5 + 4 + 3 + 2 + 1 = 36$

$7^2 = 1 + 2 + 3 + 4 + 5 + 6 + \mathbf{7} + 6 + 5 + 4 + 3 + 2 + 1 = 49$

$8^2 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + \mathbf{8} + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 64$

$9^2 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \mathbf{9} + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 81$

$10^2 = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \mathbf{10} + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 100$

and so on...

We can also get squares of natural numbers by taking the sum of first "k" odd natural numbers, where "k" is that natural number whose square is needed.

$1^2 = 1 = 1$
$2^2 = 1 + 3 = 4$ (sum of first two odd numbers)
$3^2 = 1 + 3 + 5 = 9$ (sum of first three odd numbers)
$4^2 = 1 + 3 + 5 + 7 = 16$ (sum of first four odd numbers)
$5^2 = 1 + 3 + 5 + 7 + 9 = 25$ (sum of first five odd numbers)
$6^2 = 1 + 3 + 5 + 7 + 9 + 11 = 36$ (sum of first six odd numbers)
$7^2 = 1 + 3 + 5 + 7 + 9 + 11 + 13 = 49$ (sum of first seven odd numbers)and so on...

EXERCISE 2.2

A. Find perfect squares of the following:

1. 5 **2.** 13 **3.** 19 **4.** 39 **5.** 45
6. 58 **7.** 63 **8.** 79 **9.** 97 **10.** 108

B. Find all the perfect squares between:

1. 10 and 30 **2.** 20 and 50 **3.** 40 and 70 **4.** 70 and 90

C. Write pattern of additions for the following squares:

1. 7^2 **2.** 9^2 **3.** 15^2 **4.** 11^2 **5.** 13^2
6. 8^2 **7.** 12^2 **8.** 16^2 **9.** 20^2 **10.** 27^2

2.3 SQUARE ROOTS

To find the square root is an operation opposite to find the perfect square of a number. For example, perfect square of 4 is 16 as $4^2 = 16$, therefore square root of 16 is 4.

We use the symbol $\sqrt{\ }$ for square root. Thus, $\sqrt{16} = 4$.

Note: The square root of a positive number is a value which when multiplied by itself, gives the original number.

2.3.1 Find the square root of:

(i) a natural number (ii) a common fraction (iii) a decimal given in perfect square form, by prime factorization and division method:

We have learnt methods of prime factorization and division to find the square root of a number in the previous class. Let us practice them.

(i) Square root of a natural number

Example 1. Find the square root of the following by prime factorization and division methods: **(i)** 625 **(ii)** 1600

Solution: **(i)** Square root of 625

Prime factorization method

5	625
5	125
5	25
5	5
	1

$625 = 5 \times 5 \times 5 \times 5$

or $\sqrt{625} = \sqrt{5 \times 5 \times 5 \times 5}$

$= 5 \times 5 = 25$

Division method

	25
2	$\hat{6}\overline{25}$
+ 2	4
45	225
+ 5	225
50	0

$1 \times 1 = 1$

$2 \times 2 = 4$

$3 \times 3 = 9$

$44 \times 4 = 176$

$45 \times 5 = 225$

$46 \times 6 = 276$

So, $\sqrt{625} = 25$

Solution: **(ii)** Square root of 1600

Prime factorization method

2	1600
2	800
2	400
2	200
2	100
2	50
5	25
5	5
	1

So, $1600 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5$

or $\sqrt{1600} = \sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5}$

$= 2 \times 2 \times 2 \times 5 = 40$

Division method

	40
4	1600
+ 4	16
8	00

$2 \times 2 = 4$

$3 \times 3 = 9$

$4 \times 4 = 16$

Single '0' from the pair of zeros is shifted to square root

So, $\sqrt{1600} = 40$

(ii) The square root of a common fraction (when numerator and denominator are perfect squares)

Example 2. Find the square roots of the following by prime factorization and division methods: **(i)** $\frac{9}{16}$ **(ii)** $\frac{49}{64}$

Solution: (i) Square root of $\frac{9}{16}$

Prime factorization method

Numerator: 9

3	9
3	3
	1

$9 = 3 \times 3$

or $\sqrt{9} = \sqrt{3 \times 3} = 3$

Denominator : 16

2	16
2	8
2	4
2	2
	1

$16 = 2 \times 2 \times 2 \times 2$

$\sqrt{16} = \sqrt{2 \times 2} \times \sqrt{2 \times 2}$

$\sqrt{16} = 2 \times 2 = 4$

Division method

Numerator: 9

	3
3	$\hat{9}$
+ 3	9
6	0

$1 \times 1 = 1$

$2 \times 2 = 4$

$3 \times 3 = 9$

So, $\sqrt{9} = 3$

Denominator: 16

	3
4	$\overline{16}$
+ 4	16
8	0

or $\sqrt{16} = 4$

Therefore $\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$

Solution: (ii) Square root of $\frac{49}{64}$

Prime factorization method

Numerator: 49

7	49
7	7
	1

$49 = 7 \times 7$

$\sqrt{49} = \sqrt{7 \times 7}$

$\sqrt{49} = 7$

Denominator : 64

2	64
2	32
2	16
2	8
2	4
2	2
	1

$64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$

or $\sqrt{64} = \sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 2}$

$= 2 \times 2 \times 2 = 8$

Division method

Numerator: 49

	7
7	$\overline{49}$
+ 7	49
14	6

$\sqrt{49} = 7$

Denominator : 64

	8
8	$\overline{64}$
+ 8	64
16	0

$6 \times 6 = 36$

$7 \times 7 = 49$

$8 \times 8 = 64$

or $\sqrt{64} = 8$

Therefore $\sqrt{\frac{49}{64}} = \frac{\sqrt{49}}{\sqrt{64}} = \frac{7}{8}$

(iii) The square root of a decimal (Perfect Square Form)

Example 3. Find the square root of the following by prime factorization and division methods: (i) 0.01 (ii) 1.21 (iii) 0.64

Solution: (i) To find square root of 0.01

(a) Prime factorization method

$$0.01 = \frac{0.01}{100} = \frac{1}{100}$$

Numerator = 1, $\sqrt{1} = 1$

Denominator : 100

2	100
2	50
5	25
5	5
	1

$100 = 2 \times 2 \times 5 \times 5$

$\sqrt{100} = \sqrt{2 \times 2 \times 5 \times 5}$

$= 2 \times 5 = 10$

Therefore $\sqrt{0.01} = \sqrt{\frac{1}{100}} = \frac{\sqrt{1}}{\sqrt{100}} = \frac{1}{10} = 0.1$

(b) Division method

$$\begin{array}{r|l} & 0.1 \\ 1 & \hat{0}.\overline{01} \\ +1 & 1 \\ \hline 2 & 0 \end{array}$$

So, $\sqrt{0.01} = 0.1$

Solution: (ii) To find square root of 1.21

Prime factorization method

$$1.21 = \frac{1.21}{100} = \frac{121}{100}$$

Numerator : 121

11	121
11	11
	1

$121 = 11 \times 11$

or $\sqrt{121} = \sqrt{11 \times 11} = 11$

Denominator : 100

2	100
2	50
5	25
5	5
	1

$100 = 2 \times 2 \times 5 \times 5$

$\sqrt{100} = \sqrt{2 \times 2 \times 5 \times 5}$

$\sqrt{100} = 2 \times 5 = 10$

Therefore $\sqrt{1.21} = \sqrt{\frac{121}{100}} = \frac{\sqrt{121}}{\sqrt{100}} = \frac{11}{10} = 1.1$

Division method

$$\begin{array}{r|l} & 1.1 \\ 1 & \hat{1}.\overline{21} \\ +1 & 1 \\ \hline 21 & 021 \\ +1 & 21 \\ \hline 22 & 0 \end{array}$$

So, $\sqrt{1.21} = 1.1$

(iii) To find square root of 0.64 (Students will solve it themselves)

EXERCISE 2.3

A. Find the square root of the following by prime factorization method.

1. 961
2. 1296
3. 344569
4. 817216
5. $\frac{81}{121}$
6. $\frac{2129}{9604}$
7. $\frac{1764}{7744}$
8. $1\frac{984}{14641}$
9. 249.64
10. 0.4096
11. 2981.16
12. 131.1025

B. Find the square root of the following by division method.

1. 1029
2. 14161
3. 996004
4. 10329796
5. $\frac{169}{289}$
6. $\frac{1225}{2809}$
7. $\frac{29241}{55696}$
8. $1\frac{1089}{1936}$
9. 648.7209
10. 180.9025
11. 727.9204
12. 7613.609536

2.3.2 Find the square root of a number which is not a perfect square

All natural numbers and rational numbers are not perfect squares. For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6.4}$, $\sqrt{2.5135}$.
Division method is used to find the square root of such numbers upto required decimal places. These square roots are irrational numbers.

Example 1. Find the square root of 3 up to two decimal places.

Solution: As there is no decimal value in 3, so we put two pairs of zeros to get the required decimal places.

Therefore, $\sqrt{3}$ = 1.73
(Upto two decimal places)

	1.73	
1	$\hat{3}.\overline{00}\,\overline{00}$	
+ 1	1	26 × 6 = 156
27	200	(27 × 7 = 189)
+ 7	189	28 × 8 = 224
343	1100	342 × 2 = 684
+ 3	1029	(343 × 3 = 1029)
	71	344 × 4 = 1376

Example 2. Find the square root of 40.13 upto three decimal places.

Solution:

	6.334	
6	$\overline{40}.\overline{13}\,\overline{00}\,\overline{00}$	
+ 6	36	122 × 2 = 244
123	413	(123 × 3 = 369)
+ 3	369	124 × 4 = 496
1263	4400	1262 × 2 = 2524
+ 3	3789	(1263 × 3 = 3789)
12664	6 1100	1264 × 4 = 5056
+ 4	50656	12663 × 3 = 37989
12668	10444	(12664 × 4 = 50656)
		12665 × 5 = 63325

Therefore, $\sqrt{40.13}$ = 6.334
(upto 3 decimal places)

EXERCISE 2.4

A. Find the square root of the following upto 2 decimal places.

1. 2 **2.** 5 **3.** 13 **4.** 141 **5.** 180
6. 2.5 **7.** 152.7696 **8.** 0.0143 **9.** 23.085 **10.** 125.08

B. Find the square root of the following upto 3 decimal places.

1. 6 **2.** 8 **3.** 11 **4.** 155 **5.** 205
6. 7.3 **7.** 125.79 **8.** 1468.9 **9.** 129048.27 **10.** 3355.30

2.3.3 Use rule to determine the number of digits in the square root of a perfect square.

Number of digits in the square root of a perfect square can be calculated without finding the square root by the following rule:

Let n be number of digits in perfect square, then its square root contains: **(i)** $\frac{n}{2}$ **digits when** n **is even and**

(ii) $\frac{n+1}{2}$ **digits when** n **is odd.**

Example 1. Find the number of digits in the square root of 240100 and verify your answer.

Solution: As number of digits in 240100 is 6 which is an **even** number.

So, number of digits in the square root of 240100 will be: $\frac{n}{2} = \frac{6}{2} = 3$

Hence, $\sqrt{240100} = 490$ which has three digits.

Verification:

```
      490
   ---------
 4 | 240100
+4 | 16
   ---------
 89|  801
+9 |  801
   ---------
 98|    00
```

Example 2. Find the number of digits in the square root of 34596 and verify your answer.

Solution: As number of digits in 34596 is 5 which is **odd**. So number of digits in the square root of 34596 will be: $\frac{n+1}{2} = \frac{5+1}{2} = \frac{6}{2} = 3$

Hence, $\sqrt{34596} = 186$ which has 3 digits.

Verification:

	186
1	$\hat{3}\overline{45}\overline{96}$
+ 1	1
28	245
+ 8	224
366	2196
+ 6	2196
372	0

EXERCISE 2.5

Find the number of digits in the square roots of the following perfect squares.

1. 8649 **2.** 250000 **3.** 322624 **4.** 614656

5. 432964 **6.** 81018001 **7.** 43388569 **8.** 3474748809

9. 73547776 **10.** 33942276 **11.** 7913169936 **12.** 935566751

2.3.4 Solve real life problems involving square roots

Example 1. In a room 1089 marbles are arranged on the floor in such a way that the number of marbles in each row is the same as the number of rows. Find how many rows of marbles are there.

Solution: Since the number of marbles in each row is the same as the number of rows, we have to find the square root of 1089.

As $\sqrt{1089} = 33$

So, there are 33 rows of marbles.

	33
3	$\overline{10}\overline{89}$
+ 3	9
63	189
+ 3	189
66	0

Example 2. The area of a square field is 4225 square metres. Find the length of its side.

Solution: As area of a square = $(\text{side})^2$

So, Side $= \sqrt{\text{Area of square field}} = \sqrt{4225}$

As $\sqrt{4225} = 65$

Therefore, the length of a side = 65 metres.

	65
6	$\overline{42}\,\overline{25}$
+ 6	36
125	625
+ 5	625
130	0

Example 3. Find the least number of students:

(i) When subtracted from 124

(ii) When added to 124,

(iii) When multiplied by 124, in order to make it a perfect square

	11
1	$\overline{124}$
+ 1	1
21	24
+ 1	21
22	3

Solution: Let us find the square root of the non-perfect square number 124.

As $124 = 11^2 + 3$ (greater than 11^2 by 3)

(i) Hence the required least number is 3 which when subtracted from 124, i.e., 124 – 3 = 121, gives a perfect square (less than 12^2 (11^2) by 20)

(ii) As $124 = 12^2 - 20$, So 20 is the required least number which when added to 124 i.e. 124 + 20 = 144, gives a perfect square (12^2)

Thus 124 + 20 = 12 × 12 = 144, which is the perfect square.

(iii)

2	124
2	62
31	31
	1

Thus, 124 = 2 × 2 × (31). As 31 is not occurring in pair, so 31 is the required least number by which when 124 is multiplied i.e. 2 × 2 × 31 × 31 gives a perfect square $(2 \times 31)^2$.

EXERCISE 2.6

1. A contribution was made for a party of students. Each student contributed as much as the number of students present. The collected amount was Rs 8,464. Find the amount paid by each.

2. The area of a square field is 10816m^2. Find the length of its side.

3. In a school assembly 1129 students were standing in rows such that the number of rows are equal to the number of students in a row. Find the number of students to leave to make a complete square of rows.

4. The area of a rectangular field is 1,832 square metres. Its breadth is half of its length. Find the length of the rectangular field.

5. The area of a square field is 390625 square metres. Find the length of wire required for fixing along the sides as a fence.

2.4 CUBES AND CUBE ROOTS

2.4.1 Recognize cubes and perfect cubes

(i) Cube: The cube of a number is the number obtained by raising it to the power 3. It means the number is multiplied three times by itself.

Examples. $1 \times 1 \times 1 = 1^3 = 1$, $2 \times 2 \times 2 = 2^3 = 8$

$3 \times 3 \times 3 = 3^3 = 27$, $4 \times 4 \times 4 = 4^3 = 64$

$5 \times 5 \times 5 = 5^3 = 125$, $6 \times 6 \times 6 = 6^3 = 216$.

This means: 1 is the cube of 1, 8 is the cube of 2, 27 is the cube of 3, 64 is the cube of 4, 125 is the cube of 5 and 216 is the cube of 6.

(ii) Perfect Cube: It is a number that is the result of multiplying an integer by itself three times, or it is an integer to the third power of an integer.

Examples. $(7)^3 = 7 \times 7 \times 7 = 343$, $8^3 = 8 \times 8 \times 8 = 512$

$(9)^3 = 9 \times 9 \times 9 = 729$, $10^3 = 10 \times 10 \times 10 = 1000$

Hence 1, 8, 27, 64, 125, 216, 343, 512, 729 and 1000 are perfect cubes of 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 respectively.

Similarly, we can find perfect cubes of rational numbers, like 1.1, 2.5 etc.

Perfect cube of 1.1 is $(1.1)^3 = (1.1) \times (1.1) \times (1.1) = 1.331$

Perfect cube of 2.5 is $(2.5)^3 = (.5) \times (2.5) \times (2.5) = 15.625$.

Note: Perfect cubes of irrational numbers do not exist.

Example 1. Examine whether these are perfect cubes or not.

(i) 1728 **(ii)** $5\frac{23}{64}$ **(iii)** 13.824

Solution: We can examine by making their prime factors. First of all, we find the prime factorization as under.

2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	9
3	3
	1

(i) $1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$

$= 1728 = 2^3 \times 2^3 \times 3^3 = (2 \times 2 \times 3)^3 = 12^3$

Thus, 1728 is perfect cube of 12.

(ii) $5\frac{23}{64} = \frac{5 \times 64 + 23}{64} = \frac{320 + 23}{64} = \frac{343}{64}$

$\frac{343}{64} = \frac{7 \times 7 \times 7}{4 \times 4 \times 4} = \left(\frac{7}{4}\right)^3 = \left(1\frac{3}{4}\right)^3$

Hence, $5\frac{23}{64}$ is perfect cube of $\left(1\frac{3}{4}\right)$.

Numerator

2	13824
2	6912
2	3456
2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	9
3	3
	1

(iii) $13.824 = \frac{13.824}{1000} = \frac{13824}{1000}$

$\frac{13824}{1000} = \frac{(2\times2\times2)\times(2\times2\times2)\times(2\times2\times2)\times(3\times3\times3)}{(2\times2\times2)\times(5\times5\times5)}$

$= \frac{2^3 \times 2^3 \times 2^3 \times 3^3}{2^3 \times 5^3}$

$= \frac{(2\times2\times2\times3)}{(2^3\times5)^3} = \left(\frac{24}{10}\right)^3 = (2.4)^3$

Denominator

2	1000
2	500
2	250
5	125
5	25
5	5
	1

Hence, $13.824 = (2.4)^3$

or 13.824 is perfect cube of 2.4

EXERCISE 2.7

A. Find the perfect cubes of the following:

1. 10 **2.** 20 **3.** 30 **4.** 40 **5.** 50 **6.** 60

7. 8 **8.** 12 **9.** 15 **10.** 18 **11.** 24 **12.** 25

B. Find the cubes of the following fractions:

1. $\frac{1}{2}$ **2.** $\frac{2}{3}$ **3.** $\frac{3}{4}$ **4.** $\frac{4}{5}$ **5.** $1\frac{1}{3}$ **6.** $2\frac{1}{2}$

7. $1\frac{1}{4}$ **8.** $2\frac{1}{6}$ **9.** $1\frac{1}{10}$ **10.** $2\frac{3}{20}$ **11.** $1\frac{1}{9}$ **12.** $3\frac{1}{3}$

C. Find the cubes of the decimals:

1. 0.4 **2.** 0.9 **3.** 1.2 **4.** 1.6 **5.** 2.1 **6.** 2.5

7. 0.01 **8.** 0.02 **9.** 0.05 **10.** 1.12 **11.** 1.05 **12.** 1.01

D. Examine whether the following are perfect cubes or not.

1. 512 **2.** 729 **3.** 3,375 **4.** 1,728 **5.** 64 **6.** 125 **7.** 4,096

8. 5,832 **9.** 9,261 **10.** 27,000 **11.** 32,768 **12.** 39,304 **13.** 1.728

2.4.2 Find the cube roots of numbers which are perfect cubes.

We know that, $8 = 2 \times 2 \times 2 = (2)^3$. It means cube of 2 is 8, i.e., $2^3 = 8$ or we can say cube-root of 8 is 2.

It can be written as $\sqrt[3]{8} = 2$ or $(8)^{\frac{1}{3}} = 2$.

Similarly, we can find the cube root of any perfect cube with the help of prime factorization.

Example. Find the cube root of **1.** 1000 **2.** 216 **3.** 729

Solution: (1) $1000 = 10 \times 10 \times 10 = 10^3$

Therefore, $\sqrt[3]{1000} = 10$ or $(1000)^{\frac{1}{3}} = 10$.

Solution: (2) $216 = 2\times2\times2\times3\times3\times3 = (2\times2\times2)\times(3\times3\times3)$

$216 = 2^3\times2^3 = 6^3$

Therefore $\sqrt[3]{216} = 6$ or $(216)^{\frac{1}{3}} = 6$

(3) $729 = (3\times3\times3)\times(3\times3\times3)$

$= 3^3\times3^3 = 9^3$

So, $\sqrt[3]{729} = 9$ or $(729)^{\frac{1}{3}} = 9$

2	216
2	108
2	54
3	27
3	9
3	3
	1

3	729
3	243
3	81
3	27
3	9
3	3
	1

2.4.3 Recognize Properties of Cubes of Numbers

Following are some recognized properties of cubes of numbers with examples.

(i) ***Cube of a positive number is always positive.***
For example $2^3 = 8, 5^3 = 125.$

(ii) ***Cube of a negative number is always negative.***
For example $(-3)^3 = -27, (-6)^3 = -216.$

(iii) ***Cube of an even number is always an even number*** *e.g.,* $2^3 = 8, 4^3 = 64.$

(iv) Cube of an odd number is always an odd number,
For example $3^3 = 27, 5^3 = 125.$

(v) Cubes satisfy the distributive properties as under:
(a) multiplication and (b) division

(a) $(2\times3)^3 = 2^3\times3^3$, $(8\times9)^3 = 8^3\times9^3$ **(b)** $\left(\frac{4}{5}\right)^3 = \frac{4^3}{5^3}$, $\left(\frac{6}{7}\right)^3 = \frac{6^3}{7^3}$

(vi) ***Cube numbers are the perfect cubes.*** $7^3 = 343, \quad 9^3 = 729$
So, 343, 729 and 1728 are perfect cubes

(vii) ***Every perfect cube is a cube.***

EXERCISE 2.8

A. Find the cube roots of the following:

1. 64 **2.** 343 **3.** 1331 **4.** 512 **5.** 729

6. 74088 **7.** 3375 **8.** 13824 **9.** 15625 **10.** 35937

11. $\frac{1}{8}$ **12.** $2\frac{10}{27}$ **13.** $1\frac{61}{64}$ **14.** $1\frac{91}{125}$ **15.** $\frac{343}{729}$

B. Verify with examples that:

(i) The cube of a positive number is always a positive number.

(ii) The cube of a negative number is always a negative number.

(iii) The cube of every even number is always an even number.

(iv) The cube of every odd number is always an odd number.

C. Verify the following:

1. $(3 \times 4)^3 = 3^3 \times 4^3$ **2.** $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$ **3.** $(2 \times 5)^3 = 2^3 \times 5^3$

4. $\left(\frac{3}{4}\right)^3 = \frac{3^3}{4^3}$ **5.** $(4\times5)^3=4^3\times5^3=4^3\times5^3$ **6.** $(10\times20)^3=10^3\times20^3$

7. $\left(\frac{5}{7}\right)^3 = \frac{5^3}{7^3}$ **8.** $\left(\frac{3}{5}\right)^3 = \frac{3^3}{5^3}$ **9.** $(5\times8)^3=5^3\times 8^3$

REVIEW EXERCISE 2

1. Three options are given against each statement. Encircle the correct one.

(i) Real number is:

(a) difference of rational numbers and irrational numbers

(b) intersection of set of rational numbers and set of irrational numbers

(c) union of set of rational numbers and a set of irrational numbers

(ii) Which of the following is not true about 100?

(a) whole number (b) rational number (c) irrational number

(iii) Which one of the following is a perfect square?

(a) 121 (b) 1.21 (c) 12.1

(iv) Square of 4.9 is:

(a) 4.90 (b) 24.01 (c) 240.1

(v) $\left(\frac{3}{5}\right)^2 = ?$

(a) $\frac{9}{5}$ (b) $\frac{3}{25}$ (c) $\frac{9}{25}$

(vi) $1+2+3+4+5+6+7+8+7+6+5+4+3+2+1 = ?$

(a) 8^2 (b) 9^2 (c) 7^2

(vii) If the side length of a square is $0.6m$ then its area is:

(a) $0.36m^2$ (b) $3.6m^2$ (c) $36m^2$

(viii) $\sqrt{.04} = ?$

(a) .02 (b) 2.0 (c) 0.2

(ix) $\sqrt{1^2 \times 4^2} = ?$

(a) 4 (b) 14 (c) 41

(x) $\sqrt[3]{125} = ?$

(a) 3 (b) 4 (c) 5

(xi) $\sqrt{\frac{4}{9}} = ?$

(a) $\frac{2}{3}$ (b) $\frac{2}{9}$ (c) $\frac{4}{3}$

(xii) $\sqrt{\frac{a}{b}} = ?$

(a) $\frac{a}{b}$ (b) $\sqrt{\frac{b}{a}}$ (c) $\frac{\sqrt{a}}{\sqrt{b}}$

2. Find the number of digits in the square root of the following numbers. Also find the square root.

(i) 12,544 **(ii)** 418,609 **(iii)** 30,349,081

3. Find the square root of the following:

(i) $17\frac{128}{289}$ **(ii)** $28\frac{4}{9}$ **(iii)** $101\frac{92}{169}$

(iv) 0.053361 **(v)** 0.204304 **(vi)** 65.61

4. The area of a square field is $161604 m^2$. Find the length of one of its side.

5. Find the cube root of the following numbers.

(i) 3375 **(ii)** 1728 **(iii)** $\frac{343}{512}$ **(iv)** $1\frac{602}{729}$

SUMMARY

- The number which cannot be written in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$ is called an irrational number.
- Set of real numbers is the union of set of rational and set of irrational numbers, i.e., $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.
- A number whose decimal representation is terminating is called rational number.
- The decimal fraction in which the number of digits after the decimal point is finite, is called terminating decimal fraction.
- The decimal fraction, in which the number of digits after the decimal point is infinite, is called non-terminating.
- The product of a number by it self is known as square of that number.
- Square of a positive integer or rational number is multiplied by itself two times.
- The square root of a positive number is an other positive number whose square is the given number.
- If n is the number of digits in a perfect square, then its square root contains (i) $\frac{n}{2}$ digits if n is even (ii) $\frac{n+1}{2}$ digits if n is odd.
- Cube of a number means the number is multiplied by itself three times.
- For any rational number

(i) $\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$ (ii) $\sqrt[3]{ab} = \sqrt[3]{a} \times \sqrt[3]{b}$

UNIT 3

NUMBER SYSTEMS

3.1 NUMBER SYSTEM

The number system is used for reading, writing, counting and calculation. It is based on some characters called digits. Each number is made up of these digits.

3.1.1 Recognize the Base of a Number System

The number of digits a system uses is called its **base.** The present system of counting and computation is base ten number system. This is called decimal number system.

The basic numerals of the digits of base ten system are ten in number and are as follows:

0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

The numbers from 0 to 9 are one-digit numbers. The numbers greater than 9 and less than 100 are two-digit numbers.

Similarly, the numbers greater than 99 and less than 1000 consist of three-digits and so on.

3.1.2 Define Number System with Bases 2, 5, 8 and 10

(i) *Number System with Base 2*

A number system formed by two digits, 0 and 1 is known as a number system with base 2. It is also called **binary number system**. Binary number system is not usually used in our daily life, but it is very important number system because it is used in every type of computers. Computers store data (information) in binary numbers. Thus, binary system of numbers is of primary importance in our daily life which is the modern age of computers.

In binary system of numbers, 0 is the lowest numeral whereas 1 is the greatest numeral. In this system zero is represented as 0_2, one is represented as 1_2 and two is represented as 10_2 read as one-zero to the base 2 and three is represented as 11_2 read as one-one to the base 2.

(ii) *Number System with Base 5*

In this number system, the digits 0, 1, 2, 3 and 4 are involved. Here 0 is the lowest numeral and 4 is the greatest.

Zero is represented as 0_5, One is represented as 1_5
Two is represented as 2_5, Three is represented as 3_5
Four is represented as 4_5, Five is represented as 10_5(One-zero)
Six is represented as 11_5, we read it as one-one to the base 5.

(iii) *Number System with Base 8*

In this number system, the digits 0, 1, 2, 3, 4, 5, 6 and 7 are involved. Number system with base 8 is also known as **Octal System** of Numbers. In this system the lowest numeral is zero and is represented as 0_8. The greatest numeral in this system is seven and is represented as 7_8. The number eight is represented as 10_8 and nine is represented as 11_8.

(iv) *Number System with Base 10*

In this system the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are involved. Number system with base 10 is also known as **Decimal Number System**. Decimal Number System is the most popular number system in the world because of its multiplication and division by 10 being easier.

3.1.3 Explain Number Systems with bases 2, 5, 8 and 10

(i) *Binary Number System (Number system with base 2)*

In base two system, the smallest numeral is 0 and the greatest numeral is 1. In decimal system, the greatest numeral is 9, in which when '1' is added it becomes 10 (read as ten). In the same sense when '1' is added to '1' in base two system, we get two written as '10' and read as 'one-zero' and not ten.

Thus, in binary system: one is written as 1_2

Now $1_2 + 1_2$ = One plus one is Two which is written as 10_2,

Again, $10_2 + 1_2$ = Two plus one is Three which is written as 11_2

$11_2 + 1_2$ = Three plus one is Four which is written as 100_2

In decimal system, any number can be expressed as the sum of multiples of powers of 2.

For example: $13 = 8 + 4 + 1 = \mathbf{1} \times 2^3 + \mathbf{1} \times 2^2 + \mathbf{0} \times 2^1 + \mathbf{1} \times 2^0 = 1101_2$.

$27 = 16 + 8 + 2 + 1 = \mathbf{1} \times 2^4 + \mathbf{1} \times 2^3 + \mathbf{0} \times 2^2 + \mathbf{1} \times 2^2 + \mathbf{1} \times 2^0 = 11011_2$.

(ii) *Number system with base 5*

The base 5 number system is based on the five fundamental digits 0, 1, 2, 3, and 4 to represent the numeric values.

In base five system, the smallest numeral is 0 and the greatest numeral is 4.

In decimal system 9 + 1 = 10, but in base five system $4_5 + 1_5$ = Five which is written as 10_5. Similarly, $10_5 + 1_5 = 11_5$ (Six)

Now $11_5 + 1_5 = 12_5$ (Seven) Again $12_5 + 1_5 = 13_5$ (Eight)

And $13_5 + 1_5 = 14_5$ (Nine), $14_5 + 1_5 = 20_5$ (Ten) and $20_5 + 1_5 = 21_5$ (Eleven)

Note: Any number in decimal system can be expressed as sum of multiples of powers of 5. For example: .

(a) $87 = 75 + 10 + 2 = \mathbf{3} \times 25 + \mathbf{2} \times 5 + \mathbf{2} \times 1$

Or $87 = 75 + 10 + 2 = \mathbf{3} \times 5^2 + \mathbf{2} \times 5^1 + \mathbf{2} \times 5^0 = 322_5$

(b) $138 = 125 + 10 + 3 = \mathbf{1} \times 125 + \mathbf{0} \times 25 + \mathbf{2} \times 5 + \mathbf{3} \times 1$

or $138 = \mathbf{1} \times 5^3 + \mathbf{0} \times 5^2 + \mathbf{2} \times 5^1 + \mathbf{3} \times 5^0 = 1023_5$

(iii) *Octal Number System (Number system with base 8)*

The number system with base 8 consists of the following eight fundamental digits 0, 1, 2, 3, 4, 5, 6 and 7 to represent the numeric values.

In base eight system, the smallest numeral is 0 and the greatest numeral is 7.

As in decimal system 9 + 1 = 10 (Ten), but in octal number system $7_8 + 1_8 = 10_8$ (Eight). Read as 'one zero is base eight system'.

Similarly, $10_8 + 1_8 = 11_8$ (Nine), $11_8 + 1_8 = 12_8$ (Ten)

$12_8 + 1_8 = 13_8$ (Eleven), $13_8 + 1_8 = 14_8$ (Twelve)

$14_8 + 1_8 = 15_8$ (Thirteen), $15_8 + 1_8 = 16_8$ (Fourteen)

$16_8 + 1_8 = 17_8$ (Fifteen), $17_8 + 1 = 20_8$ (Sixteen)

Note: Any number in decimal system can be expressed as sum of multiples of powers of 8; i.e., $135 = 128 + 7 = \mathbf{2} \times 64 + \mathbf{0} \times 8 + \mathbf{7} = \mathbf{2} \times 8^2 + \mathbf{0} \times 8^1 + \mathbf{7} \times 8^0 = 207_8$

(iv) *Decimal Number System (Number system with base 10)*

The number system with base 10 is also called decimal system. Decimal number system is most familiar to us. We use this decimal number system in our daily life and business for counting and calculations. In decimal system, we count in tens using the digits: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

In decimal number system, any integer can be expressed as sum of powers of ten. For example: $5683 = 5 \times 1000 + 6 \times 100 + 8 \times 10 + 3$.

or $5683 = \mathbf{5} \times 10^3 + \mathbf{6} \times 10^2 + \mathbf{8} \times 10^1 + \mathbf{3} \times 10^0$

EXERCISE 3.1

1. Answer the following :

(i) Define binary system of numbers. Write first eight numbers in binary system.

(ii) Define number system with base 5. Write first twelve numbers in base 5 system.

(iii) Define octal number system. Write first sixteen numbers in this system.

(iv) Define decimal number system giving some examples.

2. Express the following decimal numbers as a sum of multiples of powers of 2.

(i) 11 (ii) 15 (iii) 21 (iv) 29 (v) 37 (vi) 51 (vii) 63

3. Express the following decimal numbers as a sum of multiples of powers of 5.

(i) 13 (ii) 24 (iii) 39 (iv) 47 (v) 59 (vi) 78 (vii) 99 (viii) 104

4. Express the following decimal numbers as a sum of multiples of powers of 8.

(i) 30 (ii) 57 (iii) 78 (iv) 98 (v) 114 (vi) 225 (vii) 340

5. Express the following decimal numbers as a sum of multiples of powers of 10.

(i) 201 (ii) 319 (iii) 4075 (iv) 56970 (v) 980762

3.2 CONVERSIONS

We have learnt the number systems with bases 2, 5, 8 and 10. These number systems are all place value number systems. The numbers used in the systems can be converted from one system to another system. The simple and the best method for conversion is the successive division method. With the help of this method, we can convert a number from decimal system to another system.

3.2.1 Convert a number from decimal system to a System with base 2, 5, 8 and vice versa

(a) Conversion of numbers from Decimal System to a system with base 2, 5 and 8.

(i) *Conversion of Numbers from Decimal System to base Two system.*

Example 1. Convert 19 into binary number system.

Solution: We use successive Division Method.

In this method the given number 19 is repeatedly divided by 2, till the remainder is 1.

2	19
2	9 – 1
2	4 – 1
2	2 – 0
	1 – 0

Thus, continuing with the division of 19 by 2 and placing the remainders of each dividend on the right, unless the dividend is 1.

Then arranging the remainder numbers, starting from bottom to get the required number in the binary system. Thus $19 = 10011_2$.

Example 2. Convert 276 into base two system.

Solution: We use successive division method.

2	276
2	138 – 0
2	69 – 0
2	34 – 1
2	17 – 0
2	8 – 1
2	4 – 0
2	2 – 0
	1 – 0

Explanation: Continue dividing 276 by 2. Place remainder of each dividend on the right, unless the dividend is 1. Arrange the remainder numerals, starting from bottom to get the required number in base two system. Hence $276 = 100010100_2$

(ii) *Conversion of Numbers from Decimal system into Base Five system.*

Example 1. Convert 678, into base five system.

Solution: We use Successive Division Method.

In this method the given number 678 is repeatedly divided by 5, till the dividend is 4 or less than 4.

Now continue with the division of 678 by 5. Place remainder of each dividend on the right, unless the dividend is 4 or less than 4.

5	678
5	135 – 3
5	27 – 0
5	5 – 2
	1 – 0

Arrange the remainder numbers starting from bottom to get the required number in base five system. Hence $678 = 10203_5$

Example 2. Convert 6065 into base five system.

5	6065
5	1213 – 0
5	242 – 3
5	48 – 2
5	9 – 3
	1 – 4

Solution: Thus, $6065 = 143230_5$.

(iii) *Conversion of Numbers from Decimal System into Octal Number System.*

Example 1. Convert 728, into octal number system.

Solution: We use Successive Division Method. In this method the given number 728 is repeatedly divided by 8, till the dividend is 7 or less than 7.

8	728
8	91 – 0
8	11 – 3
	1 – 3

Now continue the division of 728 by 8. Place remainder of each dividend on the right, unless the dividend is 7 or less than 7. Arrange the remainder numbers, starting from bottom to get the required number in base eight system.

Thus, $728 = 1330_8$.

Example 2. Convert 2064 into octal system of numbers.

8	2064
8	258 – 0
8	32 – 2
	4 – 0

Solution: Thus, $2064 = 4020_8$.

EXERCISE 3.2

A. Convert the following decimal numbers into binary system.

1. 36	**2.** 77	**3.** 89	**4.** 156	**5.** 280
6. 489	**7.** 654	**8.** 786	**9.** 999	**10.** 1600
11. 1705	**12.** 1808	**13.** 1096	**14.** 2001	**15.** 2020

B. Convert the following decimal numbers into base five system.

1. 87	**2.** 98	**3.** 169	**4.** 205	**5.** 370
6. 444	**7.** 666	**8.** 1072	**9.** 2007	**10.** 3333
11. 5005	**12.** 5050	**13.** 5016	**14.** 5602	**15.** 5505

C. Convert the following decimal numbers into octal system.

1. 504	**2.** 1765	**3.** 20093	**4.** 33661	**5.** 48360
6. 50607	**7.** 61278	**8.** 79118	**9.** 80877	**10.** 99099
11. 90009	**12.** 99009	**13.** 98765	**14.** 98065	**15.** 99999

(b) Conversion from other Number Systems to Decimal Number System

(i) *Conversion of numbers from Binary System to Decimal System:*

Conversion of a number from binary system to decimal system is explained with the help of following examples:

Example 1. Convert **(a)** 11_2 **(b)** 111_2 **(c)** 110101_2 into decimal system

Solution:

(a) $11_2 = 1 \times 10_2^{\,1} + 1 \times 10_2^{\,0} = 1 \times 2^1 + 1 \times 2^0 = 2 + 1 = 3$

(b)
$$\begin{aligned} 111_2 &= 1 \times 10_2^{\,2} + 1 \times 10_2^{\,1} + 1 \times 10_2^{\,0} \\ &= 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 4 + 1 \times 2 + 1 \times 1 \\ &= 4 + 2 + 1 = 7 \end{aligned}$$

Hence $111_2 = 7$

(c)
$$\begin{aligned} 110101_2 &= 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 32 + 1 \times 16 + 0 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 \\ &= 32 + 16 + 0 + 4 + 1 = 53. \end{aligned}$$

(ii) *Conversion from Five Base System to Decimal Number System*

Example 1. Convert 42_5 into decimal form.

Solution: $42_5 = 4 \times 5^1 + 2 \times 5^0 = 4 \times 5 + 2 \times 1 = 20 + 2 = 22$

Example 2 . Convert 234_5 into decimal form.

Solution: $234_5 = 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0 = 2 \times 25 + 3 \times 5 + 4 \times 1$

$= 50 + 15 + 4 = 69.$

Example 3. Convert 3044_5 into decimal number system.

Solution: $3044_5 = 3 \times 5^3 + 0 \times 5^2 + 4 \times 5^1 + 4 \times 5^0$

$= 3 \times 125 + 0 \times 25 + 4 \times 5 + 4 \times 1$

$= 375 + 0 + 20 + 4 = 399$ Thus, $3044_5 = 399$.

(iii) *Conversion of numbers from octal number system into decimal number system*

Example 1. Convert 506_8 into its decimal form.

Solution: $506_8 = 5 \times 8^2 + 0 \times 8^1 + 6 \times 8^0 = 5 \times 64 + 0 \times 8 + 6 \times 1$

$= 320 + 0 + 6 = 326$ Thus, $506_8 = 326$.

Example 2. Convert 1456_8 into decimal number.

Solution: $1456_8 = 1 \times 8^3 + 4 \times 8^2 + 5 \times 8^1 + 6 \times 8^0$

$= 1 \times 512 + 4 \times 64 + 5 \times 8 + 6 \times 1 = 512 + 256 + 40 + 6$

$= 814$ Thus, $1456_8 = 814$.

EXERCISE 3.3

A. Convert the following binary numbers into decimal numbers.

1. 1101_2	**2.** 11011_2	**3.** 10111_2	**4.** 110101_2
5. 101011_2	**6.** 1110111_2	**7.** 1100110_2	**8.** 1100111_2
9. 11010110_2	**10.** 11111101_2	**11.** 10010010_2	**12.** 11000101_2

B. Convert the following five base numbers into decimal number system

1. 21_5	**2.** 30_5	**3.** 40_5	**4.** 12_5	**5.** 22_5
6. 131_5	**7.** 442_5	**8.** 232_5	**9.** 401_5	**10.** 430_5
11. 1310_5	**12.** 4421_5	**13.** 2322_5	**14.** 4013_5	**15.** 4304_5

C. Convert the following octal numbers into decimal numbers.

1. 47_8	**2.** 16_8	**3.** 40_8	**4.** 60_8	**5.** 70_8
6. 124_8	**7.** 242_8	**8.** 304_8	**9.** 450_8	**10.** 700_8
11. 3006_8	**12.** 1054_8	**13.** 6600_8	**14.** 4405_8	**15.** 5043_8

3.2.2 Add, Subtract and Multiply two Numbers with bases 2, 5 and 8

A. Binary System (Numbers with base 2)

(i) *Addition of Numbers in Binary System*

Addition in binary system is very simple because only three pairs of numbers are formed for the addition.

$0_2 + 0_2 = 0_2$

$0_2 + 1_2 = 1_2 + 0_2 = 1_2$

$1_2 + 1_2 = 10_2$

Table of Addition

+	0	1
0	0	1
1	1	10

Note: The following result is obtained, if 1 is to be added thrice.

$1_2 + 1_2 = 10_2$ Then $1_2 + 1_2 + 1_2 = 10_2 + 1_2 = 11_2$

Example 1. Find the sum of 111_2 and 101_2

Solution: First write the given numbers according to the place value of the digits below each other.

Starting from the right, add 1 to 1 to get the sum 2 which is equivalent to 10 in binary system.

Write 0 and carry 1 to add it to 1 and 0, the result is again 10. Write 0 and carry 1, add it to 1 and 1 to get 3 which in binary system is 11.

Hence, $111_2 + 101_2 = 1100_2$.

$$\begin{array}{r} 111_2 \\ +\ 101_2 \\ \hline \\ \hline \end{array}$$

$$\begin{array}{r} 111_2 \\ +\ 101_2 \\ \hline 1100_2 \\ \hline \end{array}$$

Example 2.

Add 1010_2 and 1111_2.

Solution:

$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 1010_2 \\ +\ 1111_2 \\ \hline 11001_2 \\ \hline \end{array}$$

Thus,
$1010_2 + 1111_2 = 11001_2$.

Example 3.

Add 1100_2, 1111_2 and 1001_2.

Solution:

$$\begin{array}{r} {\scriptstyle 1\ 1\ 1} \\ 1100_2 \\ 1111_2 \\ +\ 1001_2 \\ \hline 100100_2 \\ \hline \end{array}$$

Thus,
$1100_2 + 1111_2 + 1001_2 = 100100_2$.

(ii) *Subtraction of Numbers in Binary System*

The operation of subtraction in binary system is just like addition.

Example 1. Subtract:

(a) 10_2 from 101_2

Solution:

$$\begin{array}{r} 101_2 \\ -\ 10_2 \\ \hline \\ \hline \end{array} \quad \text{or} \quad \begin{array}{r} 1^{1}01_2 \\ -\ 10_2 \\ \hline 11_2 \\ \hline \end{array}$$

Thus, $101_2 - 10_2 = 11_2$.

(b) 101_2 from 1011_2

Solution:

$$\begin{array}{r} 1011_2 \\ -\ 101_2 \\ \hline 110_2 \\ \hline \end{array}$$

Or $1111_2 - 101_2 = 110_2$

Thus, $1011_2 - 101_2 =$

(iii) *Multiplication of Numbers in Binary System*

Operation of multiplication in binary system is simpler than addition, only care of position is to be taken. In multiplication three pair of numbers are obtained as shown below.

$0 \times 0 = 0$

$0 \times 1 = 1 \times 0 = 0$

$1 \times 1 = 1$

Table of Multiplication

×	0	1
0	0	0
1	0	1

Example 1.

Multiply 1101_2 and 101_2.

Solution:

Here is the process:

$$\begin{array}{r} 1101_2 \\ \times\ 101_2 \\ \hline 1101 \\ 00000 \\ 110100 \\ \hline 1000001_2 \\ \hline \end{array}$$

Hence, $1101_2 \times 101_2 = 1000001_2$

Example 2.

Solve $11100_2 \times 110_2$

Solution:

$$\begin{array}{r} 11100_2 \\ \times\ 110_2 \\ \hline 00000 \\ 111000 \\ 1110000 \\ \hline 10101000_2 \\ \hline \end{array}$$

Hence, $11100_2 \times 110_2 = 10101000_2$

EXERCISE 3.4

A. Add the following in binary number system.

1. $110_2 + 11_2$ **2.** $1010_2 + 101_2$ **3.** $1110_2 + 111_2$

4. $11011_2 + 11001_2$ **5.** $10101_2 + 110_2$ **6.** $111_2 + 11_2$

7. $1101_2 + 110_2 + 1001_2$ **8.** $1101_2 + 1001_2 + 11_2$

9. $11011_2 + 110101_2 + 110_2$ **10.** $11111_2 + 1010_2 + 111_2$

11. $1010_2 + 1111_2 + 10111_2$ **12.** $1111_2 + 11101_2 + 11111_2$

B. Solve the following in binary number system.

1. $1101_2 - 11_2$ **2.** $111_2 - 101_2$ **3.** $1110_2 - 110_2$

4. $11111_2 - 1010_2$ **5.** $110111_2 - 1101_2$ **6.** $10011_2 - 101_2$

7. $1001_2 - 10_2$ **8.** $1111_2 - 1000_2$ **9.** $11011_2 - 1010_2$

10. $11111 - 1010$

C. Simplify the following:

1. $(1111_2 + 101_2) - (1100_2 - 101_2)$ **2.** $(1101_2 + 10010_2) - (10101_2 - 1010_2)$

3. $1111_2 - 1010_2 + 101_2$ **4.** $(10101_2 - 10010_2) - (10001_2 - 1110_2)$

D. Simplify:

1. $101_2 \times 11_2$ **2.** $110_2 \times 101_2$ **3.** $1111_2 \times 100_2$

4. $110_2 \times 1101_2$ **5.** $101_2 \times 111_2$ **6.** $10101_2 \times 1010_2$

7. $11110_2 \times 101_2$ **8.** $1001_2 \times 101_2$ **9.** $10100_2 \times 11_2$

10. $110111_2 \times 111_2$ **11.** $101010_2 \times 101_2$ **12.** $101110_2 \times 110_2$

B. Base Five System

(i) *Addition and Subtraction of Numbers in Base Five System*

We know that the numerals 0, 1, 2, 3, 4, are used in base five system. The method of addition and subtraction is explained by the following examples.

Example 1. Solve $4_5 + 1_5$

Solution: The sum of four and one is five.
But five in base five system is denoted by "10_5"
Hence, $4_5 + 1_5 = 10_5$.

$$\begin{array}{r} 4_1 \\ +\ 1_5 \\ \hline 10_5 \end{array}$$

Example 2. Solve $3_5 + 4_5$.

Solution: The sum of 3 and 4 is 7 which is written as "12" in base five system.

Hence, $3_5 + 4_5 = 12_5$.

$$\begin{array}{r} 3_5 \\ +\ 4_5 \\ \hline 12_5 \end{array}$$

Example 3. Add 123_5 and 312_5

Solution:

$$\begin{array}{r} 123_5 \\ +\ 312_5 \\ \hline 440_5 \end{array}$$

or $123_5 + 312_5 = 440_5$

Example 4. Find the sum of 4141_5 and 3421_5

Solution:

$$\begin{array}{r} \overset{1}{4}\overset{1}{1}41_5 \\ +\ 3421_5 \\ \hline 13112_5 \end{array}$$

or $4141_5 + 3421_5 = 13112_5$

In order to make the operation of addition easier, students may look at the addition table of numbers in base five system.

Addition Table for Base 5 System

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

We adopt the same method in base five for the operation of subtraction as in binary system.

Example 5.

Subtract 231_5 from 424_5

Solution:

$$\begin{array}{r} 424_5 \\ -\ 231_5 \\ \hline 143_5 \end{array}$$

Or $424_5 - 231_5 = 143_5$

Explanation: We get 3 after subtracting 1 from 4, but we cannot subtract 3 from 2. Borrow 1 from 4 (leaving 3), where 1 is 5. Adding to 2 becomes 7, from which 3 is subtracted to get the difference 4. Lastly, 2 is subtracted from 3 and the difference is 1.

Example 6. Subtract 4141_5 from 13012_5

Solution:

$$\begin{array}{r} 13012_5 \\ -\ 4141_5 \\ \hline 3321_5 \end{array}$$

Or $13012_5 - 4141_5 = 3321_5$.

(ii) *Multiplication of Numbers in Base Five System*

In fact to multiply a number is the same as it is repeated addition. We adopt the same sense (order) under operation of multiplication of numbers in base five system.
This is the table of multiplication numbers in base five system.

Table of Multiplication for Base 5 System

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Examples:

$2_5 \times 2_5 = 4_5$
$2_5 \times 3_5 = 11_5$
$4_5 \times 4_5 = 31_5$
$3_5 \times 3_5 = 14_5$
$3_5 \times 4_5 = 22_5$
$10_5 \times 4_5 = 40_5$

Example 1.

Find the product of 231_5 and 34_5.

Solution:

$$\begin{array}{r} 231_5 \\ \times\ 34_5 \\ \hline 2024 \\ +\ 12430 \\ \hline 20004_5 \end{array}$$

Hence, $231_5 \times 34_5 = 20004_5$.

Example 2.

Multiply 4321_5 and 123_5

Solution:

$$\begin{array}{r} 4321_5 \\ \times\ 123_5 \\ \hline 24013 \\ 141420 \\ +\ 432100 \\ \hline 1203033_5 \end{array}$$

Hence, $4321_5 \times 123_5 = 1203033_5$.

EXERCISE 3.5

A. Solve the followings:

1. $432_5 + 134_5$
2. $3344_5 + 1304_5$
3. $4333_5 + 343_5$
4. $43_5 + 14_5$
5. $34321_5 + 24113_5$
6. $43_5 + 143_5$
7. $1432_5 + 2341_5 + 123_5$
8. $222_5 + 3433_5 + 142_5$
9. $4443_5 + 343_5$
10. $12340_5 + 3444_5 + 444_5$
11. $12340_5 + 12340_5$
12. $43324_5 + 1243_5$

B. Solve the following and check your answer.

1. $342_5 - 12_5$ **2.** $202_5 - 14_5$ **3.** $3214_5 - 1403_5$ **4.** $1203_5 - 134_5$
5. $4321_5 - 1234_5$ **6.** $4231_5 - 2304_5$ **7.** $4310_5 - 3421_5$ **8.** $20102_5 - 1424_5$

C. Solve the following:

1. $342_5 + 444_5 - 341_5$ **2.** $1234_5 + 4321_5 - 444_5$ **3.** $1321_5 + 223_5 - 44_5$
4. $44444_5 - 321_5 - 444_5$ **5.** $23222_5 - 2111_5 - 232_5$ **6.** $4304_5 - 222_5 - 333_5$
7. $1234_5 + 4321_5 - 3221_5$ **8.** $304021_5 - 103002_5 - 200403_5$

D. Find the products of the following:

1. $423_5 \times 321_5$ **2.** $14_5 \times 13_5$ **3.** $1134_5 \times 223_5$
4. $3221_5 \times 443_5$ **5.** $233_5 \times 343_5$ **6.** $403_5 \times 243_5$
7. $1004_5 \times 113_5$ **8.** $4004_5 \times 303_5$ **9.** $2410_5 \times 432_5$
10. $3210_5 \times 114_5$ **11.** $3344_5 \times 1230_5$ **12.** $4321_5 \times 234_5$

C. Octal number system (Numbers with base 8 system)

(i) *Addition in Octal Numbers System:*

We know that the numerals 0, 1, 2, 3, 4, 5, 6 and 7 are used in base eight system. While adding, if the sum of two or more digits is greater than 7, we divide the sum by 8, then write the remainder and carry the quotient to the next digit.

Example 1. Add the following octal numbers.

(i) 5_8 and 7_8 (ii) 64_8 and 74_8

Solution:

(i) $5_8 + 7_8 = \square$

Thus, $5_8 + 7_8 = 14_8$.

$$\begin{array}{r} 5_8 \\ +\ 7_8 \\ \hline 14_8 \end{array}$$

(ii) $64_8 + 74_8 = \square$

Thus, $64_8 + 74_8 = 160_8$.

$$\begin{array}{r} \overset{1}{} \\ 64_8 \\ +\ 74_8 \\ \hline 160_8 \end{array}$$

The addition table will be helpful in finding the sums in the number system with base 8.

Example 2. Solve :

(i) $445_8 + 443_8 = \square$

Solution:

$$\begin{array}{r} {\scriptstyle ①①} \\ 445_8 \\ +\ 443_8 \\ \hline 1110_8 \\ \hline \end{array}$$

Thus, $445_8 + 443_8 = 1110_8$

$\square$

$$\begin{array}{r} {\scriptstyle ①①} \\ 675_8 \\ +\ 756_8 \\ \hline 1653_8 \\ \hline \end{array}$$

Table of Addition in Octal System

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

(ii) *Subtraction:*

In subtracting one number from the other with base 8, if the digit to be subtracted is greater than the corresponding digit, we borrow '1' from the next column which is equal to 8. We add 8 to this number, then subtract.

Example 1. Evaluate the following.

(i) $14_8 - 6_8$ (ii) $604_8 - 247_8$

Solution: (i) $14_8 - 6_8 = \square$

$$\begin{array}{r} 14_8 \\ -\ 6_8 \\ \hline 6_8 \\ \hline \end{array}$$

Thus, $14_8 - 6_8 = 6_8$

(ii) $604_8 - 247_8 = \square$

$$\begin{array}{r} 604_8 \\ -\ 247_8 \\ \hline 335_8 \\ \hline \end{array}$$

Thus, $604_8 - 247_8 = 335_8$

Example 2. Subtract:

(i) 2547_8 from 3756_8 (ii) 476053_8 from 567442_8.

Solution:

(i) Subtract 2547_8 from 3756_8

Write $3756_8 - 2547_8$ in the vertical form. Then subtract.

$$\begin{array}{r} 3756_8 \\ -\ 2547_8 \\ \hline 1207_8 \end{array}$$

Thus, $3756_8 - 2547_8 = 1207_8$

(ii) Subtract 476053_8 from 567442_8

Write $567442_8 - 476053_8$ in the vertical form. Then subtract.

$$\begin{array}{r} 567442_8 \\ -\ 476053_8 \\ \hline 71367_8 \end{array}$$

Thus, $567442_8 - 47053_8 = 71367_8$

(iii) *Multiplication in Octal Number System*

This multiplication table in octal system will help for solving products.

Example 1.

Solve: $36_8 \times 43_8$

Solution:

$36_8 \times 43_8 =$ ☐

$$\begin{array}{r} {\scriptstyle ③\ ②} \\ 36_8 \\ \times\ 43_8 \\ \hline 132_8 \\ +\ 1700_8 \\ \hline 2032_8 \end{array}$$

Thus, $36_8 \times 43_8 = 2032_8$

Multiplication in Octal Number System

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Example 2. Evaluate the following:

(i) $446_8 \times 213_8$ (ii) $30456_8 \times 234_8$.

Solution:

(i) Writing $446_8 \times 213_8$ in vertical form.

$$\begin{array}{r} 446_8 \\ \times\ 213_8 \\ \hline 1562 \\ 4460 \\ +\ 111400 \\ \hline 117642_8 \\ \hline \end{array}$$

Thus, $446_8 \times 213_8 = 117642_8$

(ii) Write $(30456)_8 \times (234)_8$ in vertical form.

$$\begin{array}{r} 30456_8 \\ \times\ 234_8 \\ \hline 142270 \\ 1116120 \\ +\ 6113400 \\ \hline 7374010_8 \\ \hline \end{array}$$

Thus, $30456_8 \times 234_8 = 7374010_8$

EXERCISE 3.6

1. Add the following:

(i) $356_8 + 67_8$ (ii) $5034_8 + 6721_8$ (iii) $5003_8 + 66644_8$
(iv) $73564_8 + 62560_8$ (v) $657546_8 + 46701_8$

2. Evaluate the following:

(i) $73_8 - 46_8$ (ii) $200_8 - 173_8$ (iii) $4326_8 - 3754_8$
(iv) $77601_8 - 67706_8$ (v) $163732_8 - 77766_8$

3. Evaluate the following:

(i) $67_8 \times 45_8$ (ii) $345_8 \times 171_8$ (iii) $13470_8 \times 563_8$
(iv) $30076_8 \times 324_8$ (v) $763541_8 \times 4061$

3.2.3 Addition, Subtraction and Multiplication of Numbers with Different Bases

Let us consider the following examples.

Example 1. Simplify $12_{10} + 12_5 + 110_2$ and express the answer into binary system.

Solution: $12 + 12_5 + 110_2 = ?$

Let us convert 12_5 and 110_2 into decimal system and then simplify:

$12_5 = 1 \times 5^1 + 2 \times 5^0$

$= 1 \times 5 + 2 \times 1 = 5 + 2$

or $12_5 = 7$

and $110_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$

$= 1 \times 4 + 1 \times 2 + 0 \times 1 = 4 + 2$

or $110_2 = 6$

Now $12 + 12_5 + 110_2 = 12 + 7 + 6 = 25.$

On simplification, the result is 25.

Hence, $12 + 12_5 + 110_2 = 25 = 11001_2$

Now converting the number 25 into binary system:

2	25
2	12 — 1
2	6 — 0
2	3 — 0
	1 — 1

or $25 = 11001_2$

Example 2. Simplify $125 - 114_5 - 110_2$ and write the answer into a system with base 5.

Solution: Let us convert 114_5 and 110_2 into decimal system and then simplify.

Let us convert 114_5 and 110_2 into decimal system and then simplify:

$114_5 = 1 \times 5^2 + 4 \times 5^0$

$= 1 \times 25 + 1 \times 5 + 4 \times 1 = 25 + 5 + 4 = 34$

$110_2 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$

$= 1 \times 4 + 1 \times 2 + 0 \times 1 = 4 + 2 = 6$

Hence, $125 - 114_5 - 110_2 = 125 - 34 - 6$

$= 125 - 40 = 85$

Hence, $125 - 114_5 - 110_2 = 85 = 320_5$

Now converting the above result into a system with base 5.

5	85
5	17 — 0
	3 — 2

or $85 = 320_5$

Example 3. Simplify $102_5 \times 1001_2$ and express the answer into binary and base five systems.

Solution: $102_5 \times 1001_2 = ?$

Converting these numbers into decimal system,

$102_5 = 1 \times 5^2 + 0 \times 5^1 + 2 \times 5^0$

$= 1 \times 25 + 0 \times 5 + 2 \times 1$

$= 25 + 2 = 27$

$1001_2 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

$= 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1$

$= 8 + 1 = 9$

Now $102_5 \times 1001_2 = 27 \times 9 = 243$

Converting 243 into base five system.

Hence, $243 = 1433_5$.

5	243
5	48 — 3
5	9 — 3
	1 — 4

or $243 = 1433_5$

Again converting 243 into binary system.

2	243
2	121 — 1
2	60 — 1
2	30 — 0
2	15 — 0
2	7 — 1
2	3 — 1
	1 — 1

Therefore, $243 = 11110011_2$

Hence, $102_5 \times 1001_2 = 243 = 1433_5 = 11110011_2$.

EXERCISE 3.7

1. Simplify the following and express into binary system:

(i) $31 + 13_5 + 26_8$ (ii) $65 + 31_5 + 101_8$ (iii) $71 + 121_5 + 1010_2 + 20_8$

(iv) $34 + 34_5 + 110_2 + 15_8$ (v) $45 + 44_5 + 1010_2 + 30_8$

2. Simplify the following and express into a system with base 5:

(i) $31 - 13_5 - 101_2$ (ii) $65 - 31_5 - 10_8$ (iii) $71 - 121_5 - 111_2 - 11_8$

(iv) $100 - 100_5 - 100_2$ (v) $1000 - 1001_5 - 1010_2 - 100_8$

3. Simplify the following and write into base five and binary systems:

(i) $32 \times 114_5 \times 110_2$ (ii) $32 \times 101_5 \times 111_2$ (iii) $95 \times 1101_5 \times 101_8$

(iv) $25 \times 24_5 \times 110_2 \times 118_8$ (v) $51 \times 110_2 \times 101_5 \times 100_8$

4. Subtract the sum of 11100_2 and 101_5 from the sum of 39 and 3145 and then express the result into binary, base five and base eight systems.

5. Subtract the product of 123 and 11011 from the product of 29 and 2145 then express the result into binary, base five and base eight systems.

REVIEW EXERCISE 3

1. Convert following into decimal numbers and base eight system.

(i) 3421_5 (ii) 44332_5 (iii) 1020304_5.

2. Simplify:

(i) $12431_8 + 34211_5 + 101_2 + 289$ (ii) $40044_8 - 202023_5 - 101000_2$

(iii) $4021_8 \times 1204_5$ (iv) $11000_2 \times 1100_5 \times 100_8$.

3. Simplify and then write the result into binary, base five and base eight systems:

(i) $408 + 4310_5 + 1011_2 + 110_8$ (ii) $753_8 - 4401_5 - 111101_2$

(iii) $3412_5 \times 25_8 \times 110_2$ (iv) $25 \times 24_8 \times 23_5 \times 100_2$.

4. Fill in the blanks:

	Decimal Number	Number in base 5	Number in base 8
(i)	40	_____	_____
(ii)	82	_____	_____
(iii)	_____	132	_____
(iv)	_____	_____	62
(v)	_____	_____	705

5. In 31204

(i) The place value of 2 is 200 in _____ base system.

(ii) The place value of 2 is 128 in _____ base system.

(iii) The place value of 2 is 50 in _____ base system.

(iv) The place value of 1 is 125 in _____ base system.

(v) The place value of 1 is 512 in _____ base system.

(vi) The place value of 1 is 1000 in _____ base system.

6. A number is written as 12034: Fill in the blanks:

Place value of	In Base 5 system	In Octal system	In Decimal system
4	$4 \times 5 = 4 \times 1 = 4$	_____	_____
3	_____	$3 \times 8^1 = 24$	_____
0	$0 \times 5^2 = 0$	_____	_____
2	_____	_____	$2 \times 10^3 = 2000$
1	_____	$1 \times 8^4 =$ _____	_____

SUMMARY

- The base of decimal system is 10.
- The basic numerals or digits of decimal system are:
  0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- Every digit of a decimal number is the multiple of some power of ten.
- The number system with base 2 is also called "Binary Number System".
- In base two system, the greatest number is '1'. In binary system when one is added to one, we get 10_2.
- In base five system when 1 is added to 4, we get 10_5.
- Digits used in base five system are 0, 1, 2, 3 and 4.
- Base eight system is also called Octal Number System.
- Digits of Octal System are 0, 1, 2, 3, 4, 5, 6 and 7.
- In base eight system when 1 is added to 7, we get 10_8.
- The numbers can be converted from decimal system into binary, base five and base eight systems and vice versa.
- The binary, base five and base eight numbers can be converted into decimal system.
- We can add, subtract and multiply numbers with base 2, 5 and 8.
- We can also add, subtract and multiply numbers with different bases by converting into decimal system.
- Every digit in base five system has place value in powers of 5.
- Every digit in base eight system has place value in powers of 8.

UNIT 4

FINANCIAL ARITHMETIC

REVISION EXERCISE

1. Find the fourth proportional of the following:

   (i) 3,4,6 (ii) 7, 11,14 (iii) 38, 19, 18 (iv) 5.6, 6.3, 4.8

2. 11 and 27 are the extremes of a proportion. If one of the means is 9, find the other?

3. The cost of 25 articles is Rs 300. What will be the cost of 225 such articles?

4. How many metres of cloth will be purchased for Rs 1,315, if the cost of 40 metres of cloth is Rs 1,000?

5. A ship covered the distance of 800 knots in 40 days. In how many days will it cover the distance of 2,440 knots, if its speed remains the same?

6. A watch after every 12 hours goes fast by three minutes. How many minutes, will it go fast in 14 days?

7. The height of a tower is 20 metres and the length of its shadow is 25 metres. What is the height of another tower, whose shadow is 75 metres at the same time?

8. 120 men can do a work in 90 days. How many more men are required to complete the work in 40 days?

9. In a fort 600 persons have food provisions for 27 days and after 10 days 80 more persons joined them. For how many days will the same food provision be sufficient.

10. 200 persons have food provision for 20 days at a camp. How many persons should leave the camp so that the same food provision be sufficient for 25 days?

4.1 Compound Proportion

4.1.1 Define Compound Proportion

The relationship between two or more proportions is known as compound proportion.

There are three cases of compound proportion.

Case 1: Both proportions are direct.

Case 2: One proportion is direct and the other is inverse.

Case 3: Both proportions are inverse.

4.1.2 Solve real life problems involving compound proportion, partnership and inheritance.

A. Solve real life problems involving compound proportion

For more clarification of compound proportion, consider the following examples.

Example 1. If 70 men dig 630 m^3 of soil in 5 days, how much of the soil will 80 men dig in 10 days?

Solution: This example relates to case 1

Increase in men $\Rightarrow$ Increase in soil. ... (Direct Proportion)

Increase in days $\Rightarrow$ Increase in soil. ... (Direct Proportion)

Let x m^3 be the soil dug by 80 men in 10 days.

Men	Days	Soil (m^3)
70 ↑	5 ↑	630 ↑
80	10	x

(Both increase)

$$\Rightarrow \quad \frac{x}{630} = \frac{10}{5} \times \frac{80}{70}$$

$$\Rightarrow x = \frac{\cancel{10}^{2} \times 80 \times \cancel{630}^{9}}{\cancel{5} \times \cancel{70}} = 2 \times 80 \times 9 = 1440 \text{ m}^3.$$

Thus, 1440 m^3 soil will be dug out.

Example 2. Rs 12,000 are sufficient for a family of 4 members for 32 days. For how many days will Rs 24,000 be sufficient for a family of 8 members?

Solution: This example relates to case 2.

Increase in Rupees $\Rightarrow$ Increase in Days, i.e., Direct proportion.

Increase in Members $\Rightarrow$ Decrease in Days i.e., Inverse proportion.

Rupees		Members		Days	
12,000	↑	4	↓	32	↑
24,000		8		x	

Let x be the days for which Rs 24,000 will be sufficient for a family of 8 members.

$$\Rightarrow \quad \frac{x}{32} = \left(\frac{24000}{12000}\right) \times \frac{4}{8} = \frac{\cancel{24000}^{\,2}}{\cancel{12000}_{\,1}} \times \frac{1}{\cancel{8}_{\,1}} = \frac{1}{1}$$

$$\text{or} \quad \frac{x}{32} = \frac{1}{1} \quad \Rightarrow x = 32 \text{ days.}$$

Thus, the given amount of Rs 24000 shall be sufficient for 32 days.

Example 3. If 2,400 men have sufficient food for 20 days at the rate of 1.2 kg per person. How many men should leave so that the same food be sufficient for 40 days at the rate of 1.5 kg per person?

Solution: This example is related to case 3.

Increase in Food per person ⇒ Decrease in Men (Inverse Proportion).

Increase in Days ⇒ Decrease in Men (Inverse Proportion).

Let x be the number of men who have sufficient food for 40 days by taking 1.5 kg per person.

Food per person		Days		Men	
1.2 kg	↓	20	↓	2400	↑
1.5 kg		40		x	

$$\Rightarrow \quad \frac{x}{2400} = \frac{\cancel{20}}{\cancel{40}_{\,2}} \times \frac{\cancel{1.2}^{\,2}}{\cancel{1.5}_{\,5}} = \frac{2}{5} \qquad \text{(Both decrease)}$$

$$\Rightarrow x = \frac{2}{\cancel{5}_{\,1}} \times \cancel{2400}^{\,480} = 960 \text{ men}$$

Thus, the food will be sufficient for 960 men.

So, 2400 – 960 = 1440 men will leave.

EXERCISE 4.1

1. The fare of 12 kg of luggage for a distance of 36km is 24 rupees. How much fare will be charged for a luggage of 18kg for a distance of 12km?

2. 4800 men have sufficient food for 20 days at the rate of 1.5kg per person. Find the number of men for which the same food will be sufficient for 16 days at the rate of 1.0 kg per person.

3. The price of a carpet 16 metres long and 6 metres wide is Rs 6,288. What will be the price of 24 metres long and 12 metres wide carpet?

4. 10 men can assemble 200 cycles in 8 days. How many cycles, 5 men will assemble in 16 days?

5. If 60 persons use 40kg of sugar in 12 days, find in how many days 90 persons will use 200kg of sugar.

6. 14 persons working 5 hours daily complete a work in 8 days. Find for how many hours per day, 35 persons may work to complete the same work in 4 days?

7. A contractor engaged 30 persons to construct the roads in 30 days, but only one fourth of the work was completed after 10 days. How many more persons he requires to complete the work in time?

8. If fare for carrying out 1200 kilograms of weight at a distance of 200 kilometers is Rs 300, find how much weight can be taken to a distance of 300 kilometers for Rs 500?

9. 3 masons complete flooring 100 meters long and 40 meters broad in 16 days. In how many days will 4 masons complete the flooring 160 meters long and 50 meters broad?

10. 14 persons working 5 hours daily can complete one third of the work in 8 days. How many hours per day are needed to complete the rest of work in 4 days by 35 persons?

B. Partnership: It is a business which two or more persons run it together by sharing the profit and loss.

(i) Simple Partnership: It is the partnership in which, the partners start the business and close it together with equal or different investment capitals.

(ii) Compound Partnership: It is the partnership in which, the partners contribute different capitals for different time periods.

The profit or loss is divided on the basis of the amount of investment and the period of investment of each partner.

Solve real life problems involving partnership.

Example 1. Asad and Ammar started a business with capitals of Rs 63,000 and Rs 72,000, respectively. After one year they earned a profit of Rs 45,000. Find the share of each one in the profit.

Solution: This is the case of simple partnership. As the time period of investment of both partners is the same, so the profit will be divided on the basis of ratio of the their capital share.

Ratio between the investments in rupees:

Asad		Ammar
63,000	:	72,000
63	:	72
7	:	8

Sum of ratios = 7 + 8 = 15
Total profit = Rs. 45,000

$$\text{Asad's profit} = \frac{7}{\cancel{15}} \times \cancel{45,000}^{3,000} = \text{Rs } 21,000.$$

$$\text{Ammar's profit} = \frac{8^{1}}{\cancel{15}_{1}} \times \cancel{45,000}^{3,000} = \text{Rs } 24,000.$$

Example 2. Amjad started a business with a capital of Rs 30,000. After 6 months Umar joined him with an investment of Rs 60,000. Usman joined them after three months by investing Rs 180,000. At the end of the year they earned a profit of Rs 490,000. Find the share of each partner in the profit.

Solution: This is the case of compound partnership in which capitals and time periods of investments of partners are different.

Amjad's effective investment = Investment capital × Time period

= 30, 000 × 12 = Rs 360,000

Umar's effective investment = 60,000 × 6 = Rs 360,000

Usman's effective investment = 180,000 × 3 = Rs 540,000

Ratio of effective capitals of partners in rupees:

Amjad	:	Umar	:	Usman
360,000	:	360,000	:	540,000
36	:	36	:	54
2	:	2	:	3 (Sum of ratios = 2+2+3= 7)

Amjad's Share in profit = $\frac{2}{7}$ × 490,000 = Rs 140,000

Umar's Share in profit = $\frac{2}{7}$ × 490,000 = Rs 140,000

Usman's Share in profit = $\frac{3}{7}$ × 490,000 = Rs 210,000.

(C) Solve real life problems involving Inheritance

When a person dies, the assets left by him or her is called inheritance and it is distributed among his or her legal inheritors(heirs) according to the law of Shariah.

The steps given below will be followed:

- First of all his/her debts will be paid.
- Then as per will upto $\frac{1}{3}$ rd of his/her property will be executed, if asked for in the will.
- Then the remaining inheritance will be distributed among the inheritors as follows:

(i) If parents of the deceased person are alive, each of them will get $\frac{1}{6}$ th part of his/her property.

(ii) Widow (s) will get $\frac{1}{8}$ th part, whereas husband will get $\frac{1}{4}$ th part of the remaining property, if they have children. In case of no child, widow will get $\frac{1}{4}$ th part, whereas husband will get $\frac{1}{2}$ nd part of the remaining property (Brothers and sisters will get the rest).

(iii) Each daughter will get half of the share of each son.

Example 3. Anwer left a property of Rs 1,000,000 in cash and a plot worth Rs 600,000. If he left a widow, two sons and three daughters, how much amount will each receive?

Solution: Total property = cash + a plot

= Rs 1,000,000 + Rs 600,000 = Rs 1,600,000

Widow will receive $= \frac{1}{8} \times 1{,}600{,}000 =$ Rs 200,000

Amount left = Rs 1,600,000 – Rs 200,000 = Rs 1,400,000

Let share of a daughter be = 1, then the share of a son will be = 2

Shares of **3** daughters = 3 × 1 = 3 and Shares of **2** sons = 2 × 2 = 4

Sum of shares of children = 4 + 3 = 7.

Each son will receive $= \frac{2}{7} \times 1{,}400{,}000 =$ Rs 400,000

Each daughter will receive $= \frac{1}{7} \times 1{,}400{,}000 =$ Rs 200,000

Example 4. Mrs. Fatima died leaving behind her a property of Rs 645,000. A debt of Rs 40,000 was due to her and Rs 5,000 was spent on her burial. Distribute the amount among her mother, husband, one daughter and three sons.

Solution: The amount to be distributed among heirs

= Total amount left – Debt Amount – Burial expenses

= Rs 645,000 – Rs 40,000 – Rs 5,000 = Rs 600,000

Mother's will receive $= \frac{1}{6} \times 600{,}000 =$ Rs 100,000

Husband's will receive $= \frac{1}{4} \times 600{,}000 =$ Rs 150,000

Remaining inheritance = Rs 600,000 – Rs 100,000 – Rs 150,000

= Rs 600,000 – Rs 250,000 = Rs 350,000

Share of a daughter = 1 and share of a son = 2

Shares of 3 sons = $3 \times 2 = 6$ and sum of shares = $1 + 6 = 7$

Daughter will receive = $\frac{1}{7} \times 350{,}000$ = Rs 50,000

Each son will receive = $\frac{2}{7} \times 350{,}000$ = Rs 100,000

EXERCISE 4.2

1. Marium and Arifa started a business with investments of Rs 60,000 and Rs 80,000 respectively. After one year they earned a profit of Rs 21,000. Find the share of each partner in the profit.

2. Akram started a business with Rs 70,000. After 3 months Aslam joined the business with Rs 40,000 and after 6 months Asghar also invested Rs 40,000. At the end of the year they earned a profit of Rs 28,800. Find the share of each partner in the profit.

3. Ali, Zain and Saad started a business with Rs 30,000, Rs 30,000 and Rs 24,000, respectively. After 5 months period Zain withdraws Rs 18,000 and the business is closed after 10 months. How much did each partner receive in the profit of Rs 37,500?

4. Ali and Usman invested different amounts for different periods in a business. If the total profit in a business after one year is Rs 60,000, fill in the following blanks:

S. No.	Investment of Ali (Rs)	Investment of Usman (Rs)	Period of Ali (Months)	Period of Usman (Months)	Ratio of Profits	Profit of Ali (Rs)	Profit of Usman (Rs)
(i)	50,000	25,000	12	12	2:1	40,000	20,000
(ii)	50,000	75,000	12	12	_____	_____	_____
(iii)	25,000	25,000	6	12	_____	_____	_____
(iv)	50,000	25,000	12	6	_____	_____	_____
(v)	50,000	_____	6	6	2:3	_____	_____

5. A man left Rs 1,800,000 as inheritance. His heirs are 6 daughters and 2 sons. Find the share of each heir.

6. Aslam died leaving a property of Rs 1,660,000. He left two widows, two sons and one daughter. Find the share of each heir if the burial expenses were Rs 60,000.

7. Akram left a wealth of Rs 1,520,000. Among his heirs is a mother, a widow, three sons and four daughters. The funeral expenses are Rs 30,000 and a loan of Rs 50,000 is due to him. Calculate the share of each heir.

8. Mrs. Abdul Rehman died leaving behind her father, her mother, two sons and husband. Find the share of each son if the amount of her inheritance is Rs 480,000.

9. A person died leaving behind a number of children (No parents, no wife). If the amount of inheritance is Rs 800,000. Fill in the blanks for different number of sons and daughters.

S. No.	Number of Sons	Number of Daughters	Total Shares of Sons	Total Shares of Daughters	Amount for each Son (Rs)	Amount for each Daughter(Rs)
(i)	1	2	2	2	400,000	200,000
(ii)	2	1	4	1	_____	_____
(iii)	2	4	_____	_____	_____	_____
(iv)	_____	6	2	_____	_____	_____
(v)	3	_____	_____	4	_____	_____
(vi)	_____	_____	2	_____	_____	100,000

4.2 BANKING

The main function of a bank is to receive deposits from many of its depositors and to give loans to its account holders. In return, bank pays small amounts of profits on deposits and charges markup on loans.

4.2.1 Types Of A Bank Account

(a) Define Commercial Bank Deposits

The deposits received from customers and kept in different

accounts such as current deposit account, PLS account, foreign currency account etc. are called **commercial bank deposits**.

Following are the types of a Bank Account.

(i) PLS Saving Bank Account

This type of account encourages the account holders to deposit more amount by offering reasonable rates of profit on deposits. The term of the deposit is not fixed. Zakat is deducted on first of holy month RAMZAN.

(ii) Current Deposit Account

It is opened for managing daily transactions. Account holders are required to maintain a minimum amount of money in their accounts. Banks do not offer any profit on current accounts.

(iii) PLS Fixed Deposit Account

In this type of account, money is kept in a bank for a fixed period of time. Banks offer higher profit rates for fixed deposit accounts as compared to non-fixed deposit accounts.

(iv) Foreign Currency Account

Multinational companies or Pakistani nationals living abroad operate such type of account. In this type of account transaction in foreign currency is allowed. Zakat and tax are not deducted on this account.

(b) Describe Negotiable Instruments

It is a written promise to pay a fixed sum of money on demand or at a certain time. Common examples are cheques, pay orders and demand drafts.

(i) Cheque: It is a method of withdrawing amounts from a bank account by account holders. It is a written document addressed to a specified bank.

(ii) Demand Draft: It is an unconditional written order drawn by one bank to another bank, to pay a specified sum of money to a third party on demand.

(iii) Pay order: A negotiable instrument issued by a bank that instructs a bank to pay a certain amount of money to a third party, is called a pay order. Issuing bank is bound to pay the amount.

4.2.2 On-line Banking

(i) Explain On-line banking

On-line banking allows the customers to perform banking transactions, pay utility bills and transfer money from their account at any time, using computers and internet connection.

(ii) Transaction Through Automated Teller Machine (ATM)

Customers are provided ATM card to use as an ATM facility. The ATM software performs transactions when a customer inserts the ATM card in the machine and provides some details which include the PIN (Personal Identification Number).

(iii) Debit Card: These are provided by the bank and are used to buy goods or withdraw cash. The amount is taken from the account holder's account.

(iv) Credit Card: These are plastic cards with a magnetic strip storing information about the account holder which allows purchases locally.

4.2.3 Conversion of Currencies

Exchange rates are used to convert currencies. These rates show the relationship between the value of different currencies and are not permanent.

(a) Convert Pakistani Currency to well-known International Currencies

Foreign currency exchange rate is a price that determines how much it costs to buy the currency of a foreign country in Pakistani currency.

Table below shows some *current exchange rates of the currencies.

Country	Currency	Symbol	Buying (PKR)	Selling (PKR)
US	Dollar	$	103.40	103.65
UK	Pound	£	139.00	139.75
Saudi Arabia	Riyal	ريال	27.15	27.40
India	Rupee	₹	1.60	1.65
European (Common Market)	Euro	€	126.00	126.25
China	Yuan	¥	16.25	16.50
Japan	Yen	¥	0.946	0.953

** These exchange rates were valid at the time of writing this book (28.11.2016). These may vary from time to time and can be seen in electronic and/or print media.*

Example 1. An American wants Pakistani rupees in exchange of 250 dollars. How many rupees will he get? **(1 US dollar ($) = 103.40 PKR).**

Solution: 1 US dollar ($) = Rs 103.40

250 US dollars ($) = 250×103.40 = Rs 25,850.

Example 2. Mr. Jamal wants Saudi Riyals in exchange of 60,000 Pakistani rupees. How many Saudi Riyals will he receive?

(1 Saudi Riyal ريال = 27.15 PKR).

Solution: Amount to be converted = Rs 60,000

Rate of conversion is 1 Saudi Riyal ريال = Rs 27.15 (Pakistani)

The number of Saudi Riyals received $= \frac{60,000}{27.15} = 2,209$

EXERCISE 4.3

1. Convert:

(i) Rs 90,000 into Chinese Yuan (1 Yuan = Rs 16.25)

(ii) Pakistani Rs 50,000 into Indian Rupees

(1 Indian Rupee = Rs 1.60 Pakistani)

(iii) 110 UK Pounds into Pakistani Rupees **(1 UK Pound(£)= Rs 139 Pakistani).**

(iv) 250 Euro into Pakistani rupees **(1 Euro(€)= Rs 126.0 PKR).**

(v) Rs 64,200 Pakistani into Turkish Lira (1 Turkish Lira = Rs 42.80 Pakistani).

2. A Pakistani serving in Saudi Arabia earns 9,000 Riyals a month. He spends 2500 Riyals a month. Determine his monthly saving in Pakistani rupees, if 1 Riyal = Rs 27.15 Pakistani.

3. Kamal has Rs 38,400 (Pakistani) and 4,000 Chinese Yuans and he wants to exchange them for U.S Dollars. How many dollars will he get if the exchange rate is: 1 U.S Dollar = Rs 103.40 Pakistani and 1 Chinese Yuan(¥) = Rs 16.25 Pakistani?

4. Convert Pakistani Rs 48,000 into Indian Rupee.

5. Convert 48,000 Indian rupees into Pakistani rupees.

6. Price of a fridge in a Pakistani market is Rs 77,500. The same fridge in Japan is being sold for 70,000 Yen. Transporting expenses are Rs 5,000 and custom duty is Rs 3,000. How much will it cost if it is imported from Japan and how much money would be saved if exchange rate is Rs 0.946 = 1 Yen?

4.2.4 Profit/Markup

(i) a. Profit (I): When we deposit money into a bank, the bank uses our money and in return after some time pays an extra amount along with our actual deposit. This extra money which the bank gives us is called profit on the deposit.

(i) b. Markup (I): When we borrow money from a bank, the bank charges some extra amount along with the actual money borrowed. This extra money which the bank charges is called markup.

(ii) Principal amount (P): It is the amount which we borrow from or deposit in the bank or the amount used.

(iii) Profit/Markup Rate (R): It is the rate at which the bank gives profit on deposits or collects markup from its account holder borrowers. This rate is calculated as a percentage of the principal amount. Deposited or borrowed from the bank.

(iv) Period (T): It is the time period for which an amount is borrowed or invested.

4.2.4 (v) Calculate the profit/Markup, the principal amount, the profit or markup rate and the time period.

(A) Calculate Profit/Markup (I)

For calculation of profit or markup, we use the formula: **I = PRT**

i.e., Profit or Markup = Principal Amount × Rate × Time Period

Example 1. Akbar borrowed Rs 70,000 from a bank at the rate of 7% for 3 years. Find the amount of markup and the total amount to be paid by Akbar.

Solution: Here Principal Amount (P) = Rs 70,000,

Rate (R) = 7% Time (T) = 3 years

$\therefore$ Markup (I) = P × R × T = $70{,}000 \times \frac{7}{100} \times 3 = 700 \times 7 \times 3$ = Rs 14,700

Hence, Total Amount to be paid = Amount Borrowed + Markup

= Rs 70,000 + Rs 14,700 = Rs 84,700

(B) Calculate Principal Amount (P)

Using **I = PRT**, we get $\mathbf{P = \frac{I}{R \times T}}$

Example 2. Find the amount invested by Riaz in a business, if he receives a profit of Rs 27,000 at the rate of 12% per annum for 3 years.

Solution: Here I = Rs 27,000, R = 12%, T = 3 years

$$\therefore \quad \mathbf{P} = \frac{\mathbf{I}}{\mathbf{R} \times \mathbf{T}} = \frac{\overset{3000}{\cancel{27{,}000}} \times \overset{25}{\cancel{100}}}{\underset{1}{\cancel{12}} \times \underset{1}{\cancel{3}}} = \text{Rs } 75{,}000$$

Thus, the amount invested by Riaz = Rs 75,000

(C) Calculate Profit or Markup Rate

Using I = PRT, we get $\mathbf{R = \frac{I}{P \times T}}$

Example 3. At what annual rate percent of markup a principal amount of Rs 34,000 would become Rs 43,180 in 3 years?

Solution: Markup (I) = Total Amount – Principal Amount

= Rs 43,180 – Rs 34,000 = Rs 9,180

As, time T = 3 years; so $R = \frac{I}{P \times T} = \frac{9180}{34000 \times 3} = 0.09$

Hence, the Markup rate in percentage form = 0.09 × 100 = 9%

(D) To Calculate the Time Period (Using $\mathbf{I = PRT}$, we get $\mathbf{T = \frac{I}{P \times R}}$)

Example 4. In what time would a sum of money become triple of itself at the rate of 5% markup?

Solution: Suppose the principal amount = P = Rs 100.

Triple of Amount = 3P = Rs 300.

Markup (I) = Total Amount – Principal Amount

= 3P – P = 2P = 300 – 100 = Rs. 200

Here, $R = 5\% = \frac{5}{100} = 0.05.$

$$\therefore \quad \mathbf{T = \frac{I}{P \times R}} = \frac{200}{100 \times 0.05} = \frac{200}{5} = 40 \text{ years.}$$

Example 5. How long would Rs 25,000 have to be deposited in a bank at 12% markup per annum to receive back Rs 34,000?

Solution: Principal Amount = P = Rs 25,000, Amount Received = Rs 34,000

Markup (I) = Total Amount – Principal Amount

= 34,000 – 25,000 = Rs 9,000

As, $R = 12\% = \frac{12}{100}$, So $T = \frac{I}{P \times R} = \frac{\overset{3}{\cancel{9000}} \times \overset{1}{\cancel{100}}}{\underset{1}{\cancel{25000}} \times \underset{1}{\cancel{12}}} = 3$ years.

4.2.5 Types of Finance

Explain

(i) Over Draft (OD): It is a borrowing facility provided by the bank to account holders, businessmen, companies etc., to withdraw some amount in excess of his/her original account balance, only once.

(ii) Running Finance (RF): It is similar to over draft. It can be considered as a credit facility provided for a certain limit with a variable profit rate. It can be used again and again.

(iii) Demand Finance (DF): It is a type of loan that may be called in by the bank (or lender) at any time. It may be either for a short time or a long time.

(iv) Leasing: Lease is an agreement between the lessee (user) to pay the leaser (owner) for the use of an asset. The ownership of the leased asset during the leased period, known as term, remains with the leaser. The asset will be returned back to the lessee after agreed time.

Hire purchase is a method of buying goods in which payments of purchased price is spread over specific term by payment of an initial deposit called **down payment**. Goods will not be returned back.

4.2.5 Solve Real Life Problems Related To Banking And Finance.

Example 1. A company gets a house on lease for 6 years. According to an agreement the company paid Rs 2,000,000 as down payment and shall pay Rs 40,000 per month as rent per month. After 3 years the company shall increase the rent by 5%. Calculate total amount the leaser (owner) would get.

Solution: Down payment received by the owner = Rs 2,000,000

Rent per month for the first 3 years = Rs 40,000

$\therefore$ Rent for the first 3 years $= 3 \times 12 \times 40{,}000$ = Rs 1,440,00

Rent per month after 3 years $= 40{,}000 \times \frac{105}{100}$ = Rs 42,000

$\therefore$ Rent for next three years $= 3 \times 12 \times 42{,}000$ = Rs 1,512,000

Hence, total amount received by the owner

= Down payment + rent for first 3 years + Rent for next 3 years

= 2,000,000 + 1,440,000 + 1,512,000 = Rs 4,952,000

Example 2. The price of a car is Rs 900,000. It can be bought at 15% of the price as down payment. It had to be leased on a simple markup of 10% per year for 2 years on the remaining amount. The instalments will be made on monthly basis.

Find: (i) The monthly instalment and

(ii) The total leased price of the car to be paid.

Solution: Down payment = 15% of Rs 900,000 $= \frac{15}{\cancel{100}} \times 900{,}\cancel{000}$

$= 15 \times 9000$ = Rs 135,000

The remaining amount (P)= Rs 900,000 – Rs 135,000 = Rs 765,000

$\therefore$ Markup for 2 years on the remaining amount is given by:

$I = P \times R \times T \quad = 765000 \times \frac{10}{100} \times 2 = \text{Rs } 153{,}000$

So, the amount to be paid in 24 monthly instalments

= Remaining Amount + Markup (I) = Rs 765,000 + Rs 153000

= Rs 918,000

∴ Monthly instalment = Rs 918,000 ÷ 24 = Rs 38,250

Total Amount Paid = Down Payment+ Remaining Amount+Markup

=Rs 135,000 + Rs 765,000 + Rs 153,000 = Rs 1,053,000

EXERCISE 4.4

1. Find the profit on Rs 50,000 at the rate of 5% per year for 6 years.

2. Asad borrowed Rs 25,000 from a bank at the rate of 8% per annum for 4 years. Find the markup of the bank.

3. Naseem receives a profit of Rs 27,000 at the rate of 12% per year for 3 years. Find his original investment.

4. At what annual rate percent would Rs 68,000 amount to Rs 90,440 in 11 years?

5. How long would Rs 31,000 have to be invested at a markup rate of 6% per year to gain Rs 5,332?

6. At what rate of markup would a sum of money become double in 20 years?

7. In what time would a sum of money will double itself at 8% of markup?

8. Complete the following table:

S. No.	Markup **(I)** (Rupees)	Principal Amount **(P)** Rupees	Time **(T)** (Years)	Rate of Markup **(R)** (%)
(i)	_____	20,000	5	4
(ii)	_____	98,000	6	6.5
(iii)	10,500	21,000	_____	5
(iv)	2,100	_____	6	7
(v)	1,740	5,800	3	_____

9. Hashim buys an air-conditioner for Rs 50,000. To lease it, he has to pay 20% down payment and the remaining amount on a simple markup of 10% per year for 2½ years on monthly instalments.

Find: (i) Monthly instalment (ii) Total amount paid.

10. A company gets a house on lease for 4 years. The company paid Rs 800,000 as down payment and agreed to pay Rs 20,000 per month as rent. After 2 years the company agreed to increase the rent by 5%. Calculate the total amount that the owner (leaser) would get.

11. Arshad buys a flat for Rs 200,000 on lease. Complete the following table for different mark up rate, time period etc.

S. No.	Down Payment (Percent)	Markup rate per year (Percent)	Period of Lease (in Years)	Monthly Installment (in Rupees)	Total amount paid (in Rupees)
(i)	20%	10%	2	_____	_____
(ii)	20%	5%	2	_____	_____
(iii)	20%	10%	4	_____	_____
(iv)	40%	10%	2	_____	_____

4.3 PERCENTAGE

The percentage means "per hundred" or "out of hundred". The symbol for percentage is %. For example 40% means 40 out of 100.

4.3.1 Profit and Loss

If the selling price (S.P.) is higher than the cost price (C.P.), then the profit occurs. Profit is the difference between S.P. and C.P., i.e.,

Profit = S.P. – C.P.

If the cost price (C.P.) is higher than the selling price (S.P.), then loss occurs. Loss is the difference between C.P. and S.P., i.e.,

Loss = C.P. – S.P.

4.3.1 (i) Find percentage profit and percentage loss

Profit and loss is always on the cost price.

Hence, percentage profit or loss is expressed in terms of cost price. Thus,

$$\text{Profit percentage} = \frac{\text{Profit}}{\text{C.P.}} \times 100. \quad \text{Loss percentage} = \frac{\text{Loss}}{\text{C.P.}} \times 100.$$

Example 1. If C.P = Rs 1,000 and Profit = Rs 150, then

$$\text{Profit percentage} = \frac{\text{Profit}}{\text{C.P.}} \times 100 = \frac{150}{1000} \times 100 = 15\%$$

Example 2. If C.P = Rs 2,000 and loss = Rs 100, then

$$\text{Loss percentage} = \frac{\text{Loss}}{\text{C.P.}} \times 100 = \frac{100}{2000} \times 100 = 5\%$$

4.3.2 Discount

In order to attract customers, some times price of an article is reduced and is sold at a price lower than its marked or listed price. The difference between marked price (M.P.) and selling price (S.P.) of an article is called discount. Thus,

$$\text{Discount} = \text{Marked price} - \text{Sale price} = \text{M.P.} - \text{S.P.}$$

$$\text{Discount \%} = \frac{\text{Discount}}{\text{M.P.}} \times 100$$

Discount percentage is also known as **rate of discount**.

(ii) Find Percentage Discount

Example 1. Marked Price (M.P) = Rs 3,000, Sale Price (S.P) = Rs 2,700
Therefore, Discount M.P – S.P = Rs 3,000 – 2,700 = Rs 300.

$$\text{Discount \%} = \frac{\text{Discount}}{\text{Marked Price}} \times 100 = \frac{300}{3000} \times 100 = 10\%.$$

Example 2. S.P = Rs 1,500, Discount = Rs 200

$\therefore$ M.P = S.P+ Discount = Rs 1,500+Rs 200= Rs 1700

$$\text{Discount \%} = \frac{\text{Discount}}{\text{Marked Price}} \times 100 = \frac{200}{1700} \times 100 = 11.76\%.$$

(iii) Solve problems involving Successive Transactions (Chain Discount)

If the discounts are deducted from the marked price one after the other, it is known as chain discount (like successive transactions).

Note: Rates of discount may be different or same each time.

In chain discount, we find the amount of the first discount on the marked price and find the first S.P. Then we find the amount of second discount on the first S.P. to obtain the second S.P. and so on.

Example 1. Find the net selling price of a sofa set listed at Rs 80,000, if chain discounts of 8% and 4% are allowed.

Solution: List price (M.P.) = Rs 80,000

First discount = 8% of Rs 80,000 = $\frac{8}{100} \times 80{,}000$ = Rs 6,400

$\therefore$ First S.P. = List price – First discount = 80,000 – 6,400 = Rs 73,600

Second discount = 4% of First S.P. = $\frac{4}{100} \times 73{,}600$ = Rs 2,944.

Therefore, the second or net final S.P. = First S.P – Second discount

= 73,600 – 2,944 = Rs 70,656

Note: Students are advised to check that final S.P of the sofa will be the same, if successive discount rates are 4% and 8% (i.e., Order of discount rates does not matter).

Example 2. Asad bought a car for Rs 500,000 and sold it for Rs 550,000. Find his percentage profit.

Solution: C.P. = Rs 500,000, S.P. = Rs 550,000

$\therefore$ Profit = S.P. – C.P. = 550,000 – 500,000 = Rs 50,000

and profit % = $\frac{\text{Profit}}{\text{C.P.}} \times 100 = \frac{50{,}000}{500{,}000} \times 100 = 10\%$

Example 2. Jamal bought a house for Rs 333,000 and sold it for Rs 283,050. Find his loss percentage.

Solution: C.P = Rs 333,000, S.P. = Rs 283,050

$\therefore$ Loss = C.P. – S.P. = 333,000 – 283,050 = Rs 49,950

Hence, loss % = $\frac{\text{Loss}}{\text{C.P}} \times 100 = \frac{\overset{15}{\cancel{49{,}950}}}{\underset{1}{\cancel{333{,}000}}} \times \cancel{100} = 15\%$

Example 3. The marked price of an article is Rs 1,800. After discount, the article is sold at Rs 1,620. Find the percentage discount.

Solution: Marked price = Rs 1,800, Sale price = Rs 1,620

$\therefore$ Discount = M.P. – S.P. = 1,800 – 1,620 = Rs 180

Hence, Discount % $= \frac{\text{Discount}}{\text{Marked Price}} \times 100 = \frac{180}{1800} 100 = 10\%$

Example 4. Kamal bought some article whose marked price was Rs 5,000. He was allowed 20% discount on his purchase. Find the sale price of that article.

Solution: Marked price = Rs 5,000, Rate of Discount = 20%

$\therefore$ Discount $= 5{,}000 \times \frac{20}{100} =$ Rs. 1,000

Hence, Sale price = M.P. – Discount = 5,000 – 1,000 = Rs 4,000.

Example 5. The cost price of a toy is Rs 3,000. The shopkeeper writes the marked price (labelled) 15% above the cost price but sales it in Rs 2,760. Find the percentage discount given to the customer.

Solution: C.P. = Rs 3,000, Increase = 15% , S.P = Rs 2,700

Total increase in C.P. $= \frac{3{,}000 \times 15}{100} =$ Rs 450

Therefore, Marked price = C.P. + Total increase in C.P

= 3,000 + 450 = Rs 3,450

S.P = Rs 2,760

Discount = M.P. – S.P. = 3,450 – 2,760 = Rs 690

Percentage Discount $= \frac{\text{Discount}}{\text{Marked Price}} \times 100$

$= \frac{690}{3450} \times 100 = \frac{\cancel{690}^{1}}{\cancel{3450}_{5}} \times 100 = \frac{100}{5} = 20\%$

Example 6. A whole seller sold a motor-cycle to a retailer at a profit of 10%. The retailer sold it for Rs 37,950 at a profit of 15%. What is the cost price of the whole seller?

Solution: First of all we will find the C.P. for the retailer (which is the S.P. for the whole seller).

S.P for the retailer = Rs 37,950 Profit percentage for the retailer = 15%

Let C.P. for the retailer = Rs 100

S.P. for the retailer = C.P. + Profit % = 100 + 15 = Rs 115

If the S.P. for the retailer is Rs 115, his C.P. = Rs 100

If the S.P. for the retailer is Re 1, his C.P. $= \frac{100}{115}$

If the S.P. for the retailer is Rs 37,950, his C.P. will be:

$$\frac{100}{115} \times 37{,}950 = \frac{100}{\cancel{115}_{23}} \times \overset{330}{\cancel{37950}_{7590}} = \text{Rs } 33{,}000$$

$\therefore$ S.P. for the whole seller = C.P. for the retailer=Rs.33,000

Now we have to find the C.P. for the whole seller.

Let C.P. for whole seller = Rs 100 and % Profit of whole seller = 10%

Then S.P. for the whole seller = 100 + 10 = Rs 110

If S.P. for the whole seller is Rs 110, his C.P. = 100

If S.P. for the whole seller is Re 1, his C.P. $= \frac{100}{110}$

If S.P. for whole seller is Rs. 33,000,

then his C.P. $= \frac{100}{\cancel{110}_{1}} \times \overset{300}{\cancel{33{,}000}} = \text{Rs } 30{,}000$

$\therefore$ C.P. for the whole seller is Rs 30,000

EXERCISE 4.5

1. Complete the following table:

S. No.	C.P. in Rupees	S.P. in Rupees	Profit in Rupees	% Profit
(i)	1,050	1,155	_____	_____
(ii)	1,665	_____	333	_____
(iii)	6,000	_____	_____	22
(iv)	_____	1,250	_____	25

2. A shopkeeper purchased 160 chairs at the rate of Rs 900 per chair. He sold 60 chairs at the rate of Rs 1,000 per chair and remaining chairs at the rate of Rs 960 per chair. Find the profit or loss percentage.

3. Jalil purchased 4 old cars for Rs 900,000. He sold these cars for Rs 225,000, Rs 250,000, Rs 300,000 and Rs 215,000, respectively. Calculate the profit or loss percentage of Jalil.

4. Khalil bought a cartoon of 250 eggs for Rs 2,200. Out of these 20 eggs were rotten and 10 broken during the transportation. He sold the remaining eggs for Rs 11 each. Find the profit or loss percentage.

5. Complete the following table when marked price of an article is Rs 10,000 and is sold for different successive discounts.

S. No.	Successive Discount %	Discount (in Rupees)	Sell Price (in Rupees)	Equivalent Simple Discount %
(i)	5% and 15%	500 + 1425	9,075	19.25
(ii)	15% and 5%	_____	_____	_____
(iii)	15% and 10%	_____	_____	_____
(iv)	18% and 12%	_____	_____	_____

6. The price of a bicycle is listed as Rs 8,500. The whole seller allows the retailer chain discounts of 10% and 5%. Find S.P. Of the whole seller.

7. Find a single discount percentage equivalent to successive discounts of 20%, 10% and 5%.

8. A manufacturer sells an article which costs him Rs 5,000 at 20% profit. The purchaser sells the article at 25% gain. Find final sale price of the article.

4.4 INSURANCE

4.4.1 Define Insurance

Insurance is a symbol of protecting or safeguarding against risk or injuries. It is a contract between two parties. A person or a party

which agrees to pay an amount on monthly, quarterly or yearly basis to the insurance company is said to be **'insured'**. The insurance company provides financial protection for property, life and health losses or damages.

This contract is called **'insurance policy'**. The instalment to be paid is called **'premium'**. The time period agreed upon by both the parties is called **'maturity'**.

4.4.2 Solve Real Life Problems Regarding Life and Vehicle Insurance

(i) Life Insurance.

Life insurance is an agreement between the insured person and the insurance company for an agreed time period. Policy owner agrees to pay regular instalments of premium to the insurance company which in return agrees to pay back a sum of money at the end of agreed period or on the death or critical illness of the policy owner. The amount of premium and the time of maturity are fixed according to the age of the insurer as per rules of the company.

Example 1. A man purchased a life insurance policy for Rs 400,000. The annual premium is 4.5% of the policy amount, whereas policy fee is at the rate of 0.25%. Calculate the annual premium and quarterly premium at 27% of the annual premium.

Solution: Policy Amount = Rs 400,000

$$\text{Policy fee} = \frac{0.25}{100} \times 400000 = \text{Rs } 1000$$

$$\text{First premium} = \frac{4.5}{100} \times 400000 = \text{Rs } 18{,}000$$

$$\therefore \quad \text{Annual premium} = \text{First Premium} + \text{Policy fee}$$
$$= 18000 + 1000 = \text{Rs } 19{,}000$$

$$\text{Quarterly premium} = \frac{27}{\cancel{100}} \times 19000 = \text{Rs } 5{,}130.$$

(ii) Vehicle Insurance.

Sometimes persons or companies get insurance policies against their vehicles to cover the risk of the theft, accident, fire, etc. In

this case premium depends upon factors such as type of the vehicle, cost of the vehicle, age of the driver, etc. The premium is decided by the insurance company at different rates for different periods.

The first premium is usually the total amount of one year instalment.

Example 1. Akber got his car insured for one year, at an insurance rate of 4.5% per annum. If the price of the car is Rs 800,000, find the amount of premium.

Solution: Price of the car = Rs 800,000 and Rate of Insurance = 4.5%

$\therefore$ Amount of premium = $\frac{4.5}{\cancel{100}} \times 800,\cancel{000}$ = Rs 36,000.

Example 2. Faheem got his bus insured at a rate of 3% per annum for 2 years. The worth of his bus is Rs 4,000,000. Find the total amount paid as premium if the rate of depreciation is 10% per year.

Solution: Worth of the bus = Rs 4,000,000

Rate of annual insurance = 3%, Rate of Depreciation = 10%

$\therefore$ First premium = $\frac{3}{\cancel{100}} \times 4000\cancel{00}$ = Rs 120,000

Depreciation after one year = $\frac{10}{100} \times 4000\cancel{00}$ = Rs 400,000

Depreciated price after one year = Original price – Depreciation

= Rs 4,000,000 – Rs 400,000 = Rs 3,600,000

$\therefore$ Second premium = $\frac{3}{\cancel{100}} \times 3600\cancel{00}$ = Rs 108,000

Total amount paid as premium = First premium + Second Premium

= Rs 120,000 + Rs 108,000 = Rs 228,000

EXERCISE 4.6

1. If the amount of life premium is calculated as:

Yearly = 4.75% of the policy amount + policy fee

Policy fee = 0.2% of the policy amount

Half yearly premium = 52% of yearly premium

Quarterly premium = 27% of yearly premium
Monthly premium = 9% of yearly premium,
then complete the table below:

S. No.	Amount of Policy (Rs)	Yearly Premium (Rs) 4.75% + 0.2%	Half Yearly Premium (Rs) (52% of Yearly)	Quarterly Premium (Rs) (27% of Yearly)	Monthly Premium (Rs) (9% of Yearly)
(i)	100,000	_____	_____	_____	_____
(ii)	150,000	_____	_____	_____	_____
(iii)	300,000	_____	_____	_____	_____

2. Hameed got a life insurance policy of Rs 300,000. Find the first premium he has to pay when the rate of annual premium is 5.2% and policy fee is 0.25% .

3. Basheer insured his car for Rs 500,000, at a rate of 2% per annum for 3 years. The depreciation rate is 5% per year. Find the total amount he has to pay as premium.

4. If the amount of vehicle premium calculated yearly is 4.0% of the worth of the vehicle and Depreciation rate = 10% per annum.

Time period = 3 years, then complete the table given below:

S. No.	Worth of Vehicle (Rs in Thousands)	First Premium (Rs in Thousands)	Second Premium (Rs in Thousands)	Third Premium (Rs in Thousands)	Total Premium (Rs in Thousands)
(i)	100	4	3.60	3.24	10.84
(ii)	250	10	_____		_____
(iii)	300	_____	_____	_____	_____
(iv)	_____	_____	_____	16.20	_____
(v)	_____	_____	36	_____	_____

5. A lady insured her self at a premium rate of 5% for one year. She paid Rs 25,000 as premium. How much is the value of her insurance?

4.5 INCOME TAX

4.5.1 Explain Income Tax, Exempt Income and Taxable Income.

(a) Income Tax: Income tax is imposed by the government on the annual income of a person or company whose income exceeds a certain limit. The rules and rates for income tax are amended by the government and are announced during the presentation of annual budget.

(b) Exempt Income: It is the income which is not subject to income tax under the present income tax law. Exempted income is as follows:

(i) Agriculture Income (ii) Gratuity or pension

(iii) Income of house or property owned by widows.

(c) Taxable Income: It is the difference of total annual income and exempted income.

Taxable Income = Total Annual Income – Exempted Income.

(d) Rebate: Government announces a certain limit of earning on which a person has not to pay income tax. This limit is called rebate. For example, if annual income of a salaried person is Rs 500,000, the rebate is Rs 400,000, then he has to pay income tax only on Rs 100,000, at the rate announced every year in the budget.

(e) Assessment Period: It starts on first July every year and ends on 30th June the following year.

Taxable Income Slabs 2016 - 2017

S. No.	Taxable Income	Rate of Tax
1.	*Where the taxable income does not exceed Rs 400,000*	0%
2	*Where the taxable income exceed Rs 400,000 but does not exceed Rs 500,000*	2% of the amount exceeding Rs. 400,000
3.	*Where the taxable income exceed Rs 500,000 but does not exceed Rs 750,000*	Rs 2,000 + 5% of the amount exceeding Rs 500,000
4.	*Where the taxable income exceed Rs 750,000 but does not exceed Rs 1,400,000*	Rs 14,500 + 10% of the amount exceeding Rs 750,000

S. No.	Taxable Income	Rate of Tax
5.	*Where the taxable income exceed Rs 1,400,000 but does not exceed Rs 1,500,000*	Rs 79,500 + 12.5% of the amount exceeding Rs 1,400,000
6.	*Where the taxable income exceed Rs 1,500,000 but does not exceed Rs 1,800,000*	Rs. 92,000 + 15% of the amount exceeding Rs 1,500,000
7.	*Where the taxable income exceed Rs 1,800,000 but does not exceed Rs 2,500,000*	Rs 137,000 + 17.5% of the amount exceeding Rs 1,800,000
8.	*Where the taxable income exceed Rs 2,500,000 but does not exceed Rs 3,000,000*	Rs 259,500 + 20% of the amount exceeding Rs 2,500,000
9.	*Where the taxable income exceed Rs 3,000,000 but does not exceed Rs 3,500,000*	Rs 359,500 + 22.5% of the amount exceeding Rs 3,000,000
10.	*Where the taxable income exceed Rs 3,500,000 but does not exceed Rs 4,000,000*	Rs 472,000 + 25% of the amount exceeding Rs 3,500,000
11.	*Where the taxable income exceed Rs 4,000,000 but does not exceed Rs 7,000,000*	Rs 597,000 + 27.5% of the amount exceeding Rs 4,000,000
12.	*Where the taxable income exceed Rs 7,000,000*	Rs 1,422,000 + 30% of the amount exceeding Rs 7,000,000

4.5.2 Solve Simple Real Life Problems Related to Individual Income Tax Assesses

Example 1. Annual income of Junaid is Rs 480,000. Calculate the amount of income tax to be paid by Junaid.

Solution: Annual income = Rs 480,000

This income is listed in the Taxable income slab at Sr.# 2, where Rate of Tax is 2% of the amount exceeding Rs 400,000

Here minimum income tax is 0 (Rs 0.0)

Income exceeding Rs 400,000 is:

Rs 480,000 – Rs 400,000 = Rs 80,000

$\therefore$ Income tax = Rate of Tax × Income exceeding Rs 400,000

$= 2\% \times 80{,}000 = \frac{2}{100} \times 80{,}000 =$ Rs 1,600

Example 2. Annual income of Ashraf is Rs 900,000. Calculate the amount of income tax, if he paid Rs 50,000 as Zakat.

Solution: Annual income = Rs 900,000, Amount Paid as Zakat = Rs 50,000

$\therefore$ Income calculated for tax = 900,000 – 50,000 = Rs 850,000

This income is listed in Taxable income slab at Sr. # 4, where rate of tax is 10% of the amount exceeding Rs 750,000 which is

Rs 850,000 – Rs 750,000 = Rs 100,000

Also minimum income tax is Rs 14,500

So, Total Income Tax = Minimum Income Tax + Tax on Rs 100,000

= Rs 14500 + 10% of Rs 100,000

= 14,500 + 10,000 = Rs 24,500.

$\therefore$ Ashraf has to pay Rs 24,500 as income tax.

Example 3. Harmain paid Rs 107,000 as income tax in a year. Find her annual income.

Solution: Income tax paid in a year = Rs 107,000

As this amount is greater than Rs 92,000 by Rs, 15,000 but less a than Rs 137,000, so taxable income of Harmain falls in income slab at Sr # 6, where rate of tax is 15% of the amount exceeding Rs 1,500,000.

Thus we have to find the amount exceeding Rs 1,500,000 on which Harmain paid Rs 15,000 more than Rs 92,000 as income tax.

If Rs 15 is income tax on Rs 100 then Re 1 is income tax on Rs $\frac{100}{15}$

$\therefore$ Rs 15,000 is the income tax on Rs $\frac{100}{15} \times$ 15,000 = Rs 100,000

Hence, annual income of Harmain is: Rs 1,500,000 + Rs 100,000

= Rs 1,600,000.

Verification: Income tax on Rs 1,600,000 = Rs 92,000 + 15% of 100,000

= Rs 92,000 + 15,000 = Rs 107,000 **(Verified)**

EXERCISE 4.7

1. Monthly salary of Amjad is Rs 75,000. Compute his annual income tax.

2. Hania earns Rs 495,000 in a year. Calculate her income tax, if she paid Rs 40,000 as Zakat.

3. A person has earned Rs 7,000,000 in a year. The Tax deducted at source is Rs 120,000 and Zakat deducted is Rs 130,000. Calculate the income tax which he has to pay at the end of the financial year.

4. The monthly salary of Ismail is Rs 200,000. Find the income tax he paid, if Rs 80,000 and Rs 20,000 are paid as Zakat and wealth tax, respectively.

5. Sufyan paid Rs 497,000 as income tax in a year. Find his monthly income.

6. Bushra paid Rs 289,500 as income tax at the end of a year. If she has paid Rs 130,000 as wealth tax and Rs 120,000 as Zakat, find her annual income.

7. Complete the following table for different income categories:

S. No.	Yearly Income (Rs in 1000)	Income Tax Slab Number	Min. Income Tax (Rs)	Lower Limit (Rs)	Income Exceeding Lower Limit (Rs in 1000)	Tax Rate (%)	Total Income Tax (Rs)
(i)	578	3	2,000	500,000	78	5%	5,900
(ii)	480	_____	_____	_____	_____	_____	_____
(iii)	900	_____	14,500	_____	_____	_____	_____
(iv)	6,000	11	_____	_____	2,000	27.5%	_____

REVIEW EXERCISE 4

1. Encircle the correct answer:

(i) The concession on a written price is called:

(a) Tax (b) Profit (c) Discount (d) Loss

(ii) The relation between two or more proportions is called:

(a) Direct Proportion (b) Indirect Proportion

(c) Inverse Proportion (d) Compound Proportion

(iii) ATM stands for:

(a) Account Telly Machine (b) Account Transfer Machine

(c) Automated Teller Machine (d) Auto Cash Transfer Machine

(iv) The rate of Zakat is:

(a) 5% (b) 2.5% (c) 10% (d) 1.0%

(v) The cost price of a book is Rs. 300. If it is sold for 20% discount, then the selling price of the book is:

(a) Rs 240 (b) Rs 260 (c) Rs 230 (d) Rs 280

(vi) If the rate of conversion of 1 Saudi Riyal = Rs 27.15,

then Rs 54,300 = _____ Saudi Riyals.

(a) 1900 (b) 2,100 (c) 2,000 (d) 2,010

(vii) The tax imposed on income is called:

(a) Ushr (b) Jazya (c) Income Tax (d) Property Tax

(viii) Property of a deceased person is to be distributed among a widow, three daughters and two sons. Each son will get a share:

(a) $\frac{1}{8}$ th (b) $\frac{1}{4}$ th (c) $\frac{1}{6}$ th (d) $\frac{1}{3}$ rd

(ix) A sum of money would triple itself at 8% rate of markup in_____years:

(a) 40 (b) 30 (c) 20 (d) 25

(x) A single discount equivalent to successive discounts of 10% and 5% is:

(a) 15% (b) 14.5% (c) 15.5% (d) 14%

2. A company earns a profit which is to be divided among four partners A, B, C and D in the ratio 3 : 4 : 5 : 6. If D gets Rs 54,00, how much will A and B get? What is the total profit?

3. If 30persons use 20 kg of sugar in 50 days, find in how many days 15 persons will use 320 kg of sugar?

4. Which is the greater discount, a single discount of 25% or two successive discounts of 15% and 10% ?

5. Akber purchased a car in Rs 1,000,000 and sold it to Asgharon 20% profit. Then Asghar sold it to Anwar on 20% loss. Find the price paid by Anwar?

6. Usama paid an amount of Rs 40,500 as the first premium of one year for insurances of his car. How much is the price of his car, if the rate is 3.85% and Rs 2,000 are service charges?

7. A factory marked prices of the confectionary items at 20% above the cost price and SAD them to the confirm at some discount. The cost price of confectionary items is Rs 2,500 and the selling price is Rs 2,700. Find the discount % given to the customer.

8. Find the net annual income of Ahmad if he pays Rs 8,900 as income tax and Rs 20,000 as Zakat.

SUMMARY

- Compound proportion is the relationship between three or more proportions.
- When two or more persons run a business together it is called partnership.
- When a person dies then the assets left by him/her is called inheritance to be distributed among his/her legal heirs.
- A bank which accepts deposits, provides loans and services to the clients is known as commercial bank.
- An account on the basis of profit and loss sharing is known as a PLS account.
- Current deposit account is a running account without any interest. It is used by businessmen for managing daily transactions.
- A cheque is a written order that instructs a bank to pay the specific amount from a specified account to the holder of the cheque.
- A Demand Draft is a written order drawn by one bank to another bank, to pay a specified sum of money to the holder of DD on demand.
- Pay order is a document which instructs a bank to pay a certain amount to a third party. The bank which issues a pay order provides a guarantee that the payment will be made.
- On-line banking is the use of instructions by the banks to assist their customers to withdraw money, make deposits, transfer money, check balances, etc.
- An Automated Teller Machine (ATM) is an electric device that allows bank customers to draw cash, check their balances

(at any time) and transfer amounts from one account to other.

- Credit card is a plastic card with a magnetic strip which stores information related to the account holder in a particular bank. These are used for purchasing without cash up to certain limit.
- Markup is the extra money which a bank receives from a client on borrowed money.
- The rate at which the bank gives loan to its account holder is called mark-up rate.
- Over Draft (OD) is a facility provided by a bank to the account holder to withdraw some amount in excess of their original balance (only once).
- **I = PRT,** where P = Principal amount, R = Rate of markup T = Time period and I = Markup
- A lease is an agreement between the leasee (user) to pay some amount to leaser (owner) for the use of an asset.
- Discount means reduction in the marked price of an article Discount = Marked Price – Sale Price.
- Chain discount means giving discount again and again at various rates.
- Life insurance is an agreement between the insurance company and the policy owner for an agreed period of time and amount.
- Income tax is imposed on the amount of income of a person or firm when the income exceeds a certain limit which is determined by the government every year.
- Exempted income is the income which is not subject to income tax, (such as Agricultural income, pension, property owned by widows).
- Taxable Income = Annual Income – Exempted Income.
- Rebate is the non-taxable income.
- Assessee is a person by whom any tax is payable under the income tax law.

UNIT 5

POLYNOMIALS

5.1 ALGEBRAIC EXPRESSION

Algebra was introduced by Muslim Mathematician named Al Khawaraizmi (780-850). He was also considered the "father of modern Algebra".

A combination of variables and constants obtained by algebraic operations of addition (+), subtraction (–), multiplication ($\times$), division ($\div$), root extraction ($\sqrt{\ }$) and raising to powers, is called an algebraic expression or simply an expression.

For example, (i) x (ii) -5 (iii) xyz

(iv) $\sqrt{x} - \frac{1}{\sqrt{x}}$ (v) $\frac{1}{2}x^3 - \frac{1}{3}x^2 - \frac{1}{5}x + \frac{1}{4}$ are algebraic expressions.

Every signed part of an algebraic expression is said to be a term.

In the last example (v), $\frac{1}{2}x^3$, $-\frac{1}{3}x^2$, $-\frac{1}{5}x$ and $\frac{1}{4}$ are all terms of the expression.

In short, algebraic expressions are used for solving equations and expressing formulas to escape from the old method of writing every thing in words and making life easier by increasing abstract thinking and saving time.

5.1.1 Recall Some basic terms: Constant, Variable, Literal and Coefficient

(i) Constant: A quantity having a fixed numerical value is called a constant. For example in expressions $2x$, $3x + 9y - 1$ and $5x^2y - 4xy^2 + 8$, 2, 3, 9, –1, 5, – 4 and 8 are constants respectively.

(ii) Variable: A symbol or letter used to represent some element of a non-empty set, is called a variable or an unknown. In the above examples x and y are used as variables. Variables are denoted by small (lower case) letters x, y, z etc. of the English alphabet.

(iii) Literal: A letter used to represent a constant or a variable in an expression is known as literal.

For example: In $ax + by + c$, a, b and c are literals representing constants where as x and y are literals showing variables. Also in $2x - 5y + p$; x, y and p are literals but 2 is not a literal.

(iv) Coefficients: A coefficient is a constant which is a multiplier of the variable (or variables) in a term.

5.2 POLYNOMIAL

5.2.1 Define Polynomial, Degree and Coefficients of a polynomial

(i) Polynomial: A polynomial in one variable, say x, is an expression of the type.

$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$, where n is a non-negative integer and the coefficients $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers.

It is usually denoted by P (x):

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n \ldots (1)$.

For example, (i) 1 (ii) 0 (iii) $2y - \frac{1}{2}$ (iv) $x^2 + 2x - 3$

(v) $\sqrt{3}x^2y + 1$, are polynomials, but (vi) $\frac{3}{x}$, (vii) $6\sqrt{x-y}$,

(viii) $2x^{2/3} + 5x^{5/7}$ (ix) $\frac{1}{x^2}$ are not polynomials. (Why?)

(ii) Degree of a Polynomial: If in (1), $a_0 \neq 0$, polynomial $P(x)$ is said to be a polynomial of degree n.

If $n = 0$ and $a_0 \neq 0$, then $P(x) = a_0$ means that the polynomial is a constant. A constant is a polynomial of degree zero.

If $n = 0$ and $a_0 = 0$, then $P(x) = 0$ which is also a polynomial but no degree is associated with it. (Why?)

(iii) Coefficients of a Polynomial: We know that polynomial in one variable is $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$

Here a_0, a_1, a_2, ..., a_{n-1}, a_n are real numbers which are called coefficients of the polynomial.

For Example. In $2x^4 - 3x^3 + 7x^2 - 8x + 5$; *2, –3, 7, –8 and 5 are all coefficients of the polynomial.*

5.2.2 Recognize polynomials in one, two and more variables

Let us recognize polynomials with respect to variables by the following examples:

(i) $4x, 2x^2+9, 5a^2+6a-1$ and y^3-3y^2+y+7

Each of them is a polynomial in ONE VARIABLE.

(ii) $3xy, 2x^2-3y+2, -a^3+2b^3$ and $p^4+4p^2q-2pq^2+q^4$

are polynomials in TWO VARIABLES.

(iii) $x^2yz + xy^2z + xyz^2 + 5$ and are $xyz + xy + yz + zx$

polynomials in THREE VARIABLES x, y and z.

Thus polynomials can have one, two or more variables.

5.2.3 Recognize polynomials of various degrees

(e.g. linear, quadratic, cubic and biquadratic polynomials)

As the degree of a polynomial is the highest degree of its non-zero term, so let us recognize polynomials by their degrees.

(i) Linear Polynomial: Whose degree is 1.

Examples. $5x, 2x+3y, 5x-9y-z, \frac{x}{4}+4$ are all linear polynomials.

(ii) Quadratic polynomial: Whose degree is 2.

Examples. $xy, \frac{1}{3}y^2, 2xy+\sqrt{3}, p^2+q^2+r$ are all quadratic polynomials.

Note: Polynomial xy is having only one non-zero term, so its degree is the sum of exponents of x and y which is $1 + 1 = 2$.

(iii) Cubic Polynomial: Whose degree is 3.

Examples. $x^2y, xyz+3x+1, x^3+y^3$ and $5pq^2+2p^2+5q^2$ are all cubic polynomials.

(iv) Biquadratic Polynomial: Whose degree is 4.

Examples. $2x^4, x^3y-xy^3, x^4+x^3y-5xy^3-y^4$ and $p^2qr+pq+qr+pr+3$ are all biquadratic polynomials.

EXERCISE 5.1

1. Separate constants, variables and literals from each of the following algebraic expressions.

(i) $3x + 1$ (ii) 0 (iii) $2a^2 + \frac{1}{3}b - 3$ (iv) $3l - 2m$

(v) $a + b^2 - 2c + 4$ (vi) $px + qy + r$ (vii) $-5x + 9y - 4$ (viii) $3^2p - 2^2q$

(ix) $y^2 - y - 1$ (x) $\sqrt{3}a - 9ab + \sqrt{5}$ (xi) $7x^2 - 2x + 3$ (xii) -8

2. Identify polynomials from the following.

(i) -1 (ii) $\frac{3}{y}$ (iii) $\frac{1}{x} - x$ (iv) $\sqrt{5}x + y$

(v) $5xy^3$ (vi) $2 - x$ (vii) $\frac{\sqrt{1}}{3}$ (viii) $\frac{1}{x^2 + 3}$

(ix) $x^2 + 3x - 1$ (x) $3^5 + \frac{4}{x}$ (xi) $ax^2 + bx + c$ (xii) $2a^2 + 3a + 1$

3. Indicate the degree of the following polynomials.

(i) 5^3 (ii) x (iii) $x + y^2$ (iv) $x^2y^2 + y^2$

(v) $x^3y^2z^2 + 1$ (vi) $-\frac{1}{4} + 3x + 5$ (vii) $x^4y + y^2 + y^3$ (viii) 3^2xyz

(ix) $x^6 + x^2y^3 + xy^4$ (x) $-x^3 + 8xyz^2$ $\frac{2}{13}$ (xi) (xii) $4^2x^4 5^2y^4$

4. Write coefficients of the following polynomials.

(i) 3 (ii) $\sqrt{4}xy$ (iii) $-x$ (iv) $x^2yz + 1$

(v) $5x^2 + 9y + z$ (vi) $\sqrt{2}x^3 - \sqrt{3}y^4$ (vii) $-2xyz$ (viii) $-\frac{3}{4}x^2$

5. Find the number of variables in each of the following polynomials.

(i) $3x + 2y$ (ii) $5x^2 - 4y - 3z$ (iii) $x^2 - y^3 + 1$ (iv) $2x^2y$

(v) $xy + 3$ (vi) $3x^7$ (vii) 25 (viii) $6x^3y^4z^2$

6. Recognize and write types of the following polynomials as linear, quadratic, cubic or biquadratic.

(i) $x+2y$ (ii) $3x-5$ (iii) $2xy+\frac{y}{5}$ (iv) y^2+y-3

(v) x^3+xy+5 (vi) $x^4-2x^3+\frac{1}{2}$ (vii) $x^3+x^2+\sqrt{4}$ (viii) $5x^4$

(ix) $-9xy^2$ (x) $x^2+2x^2y^2+y^2$ (xi) x^2-x (xii) $3xy^2z^3$

5. 3 OPERATIONS ON POLYNOMIALS

5.3.1 Add, Subtract and Multiply Polynomials

We have already learnt the above operations on polynomials.

(i) Addition of Polynomials

Polynomials can be added by combining like terms using commutative, associative or distributive properties. It can be done by horizontal or vertical methods.

Example 1. Add $2x+3y-4z$ and $5z+6x-3y$ horizontally.

Solution: Required sum $= (2x+3y-4z)+(5z+6x-3y)$

$$= 2x+3y-4z+5z+6x-3y$$
$$= 2x+6x+3y-3y-4z+5z$$
$$= 8x+z$$

Example 2.

Add $a^2-ab+2bc+3c^2$, $2ab+b^2-3bc-4c^2$ and $ab-4bc+c^2-a^2$ by vertical and horizontal methods.

Solution: Vertical Method

$$\begin{array}{r} a^2 - ab + 2bc + 3c^2 + 0b^2 \\ 0a^2 + 2ab - 3bc - 4c^2 + b^2 \\ -a^2 + ab - 4bc + c^2 + 0b^2 \\ \hline +0 + 2ab - 5bc + 0 + b^2 \end{array}$$

Horizontal Method

Required sum $= (a^2-ab+2bc+3c^2)+(2ab+b^2-3bc-4c^2)+(ab-4bc+c^2-a^2)$

$=a^2-ab+2bc+3c^2+2ab+b^2-3bc-4c^2+ab-4bc+c^2-a^2$

$=a^2-a^2-ab+2ab+ab+2bc-3bc-4bc+b^2+3c^2-4c^2+c^2=2ab-5bc+b^2$

So, $2ab-5bc+b^2$ is the required **sum**.

(ii) Subtraction of polynomials

Subtraction of one polynomial from the other can be done by changing the sign of every term in the polynomial to be subtracted and then adding this result to the other polynomial.

Example 1. Subtract $2a + 3b + 4$ from $6 + 5a - 6b$

Solution: Required difference $= (6 + 5a - 6b) - (2a + 3b + 4)$

$= 6 + 5a - 6b - 2a - 3b - 4$

Example 2. Subtract $a^4 - 7a^3b + 6a^2b^2 + 5ab^3 + 6b^4$ from $5ab^3 + 6b^4 - a^4 + 7a^3b - 8a^2b^2 + 7$ by vertical and horizontal method.

Solution: **Vertical Method**

$$\begin{array}{rrrrrrrrrr} 5ab^3 & +6b^4 & - & a^4 & +7a^3b & - & 8a^2b^2 & + & 7 & \\ +5ab^3 & +6b^4 & + & a^4 & -7a^3b & + & 6a^2b^2 & - & 0 & \\ - & - & - & + & - & + & & & & \text{(change of sign)} \\ \hline 0 & +0 & - & 2a^4 & +14a^3b & - & 14a^2b^2 & + & 7 & \end{array}$$

Horizontal Method

Required Difference is $(5ab^3 + 6b^4 - a^4 + 7a^3b - 8a^2b^2 + 7)$

$$-(a^4 - 7a^3b + 6a^2b^2 + 5ab^3 + 6b^4)$$

$= 5ab^3 + 6b^4 - a^4 + 7a^3b - 8a^2b^2 + 7 - a^4 + 7a^3b - 6a^2b^2 - 5ab^3 - 6b^4$

$= 5ab^3 - 5ab^3 + 6b^4 - 6b^4 - a^4 - a^4 + 7a^3b + 7a^3b - 8a^2b^2 - 6a^2b^2 + 7$

$= -2a^4 + 14a^3b - 14a^2b^2 + 7 \Rightarrow 14a^3b - 2a^4 - 14a^2b^2 + 7$

So, $14a^3b - 2a^4 - 14a^2b^2 + 7$ is the required **difference**.

(iii) Multiplication of Polynomials

Multiplication of Polynomials is performed by using the laws of exponents, the rules of signs and the commutative, associative and the distribute properties.

For example. $2x$ multiplied by $3y$ is written as: $2x \times 3y = 6xy$

Example 1. Find the product of $3a^2bc$ and $4ab^3c^4$

Solution: $(3a^2bc)\,(4ab^3c^4)$

$= (3 \times 4)\,(a^2bc)\,(ab^3c^4)$ (Associative Law)

$= 12a^{2+1}\,b^{1+3}\,c^{1+4}$

$= 12a^3b^4c^5$ (Law of exponents)

Example 2. Multiply $x^2 - 2x - 5$ *by* $x + 3$

Solution:

Vertical Method

$$\begin{array}{r} x^2 - 2x - 5 \\ x + 3 \\ \hline x^3 - 2x^2 - 5x \\ +\,3x^2 - 6x - 15 \\ \hline x^3 + x^2 - 11x - 15 \end{array}$$

or

Horizontal Method

$(x^2 - 2x - 5)\,(x + 3)$

$= x^2\,(x + 3) - 2x\,(x + 3) - 5\,(x + 3)$

$= x^3 + 3x^2 - 2x^2 - 6x - 5x - 15$

$= x^3 + x^2 - 11x - 15$

So, the required product is $x^3 + x^2 - 11x - 15$

5.3.2 Divide a Polynomial by a Linear Polynomial

Division is the reverse process of multiplication. Let us learn this process by an example.

Example 1. Divide $6x^2 + 8x - 14$ by 2

Solution: $(6x^2 + 8x - 14) \div 2 = \dfrac{(6x^2 + 8x - 14)}{2} = \dfrac{6x^2}{2} + \dfrac{8x}{2} - \dfrac{14}{2} = 3x^2 + 4x - 7$

Example 2. For what value of k, the polynomials $x^2 + 5x + k$ is divisible by $x + 4$

Solution:

$$\begin{array}{r} x+4\;)\;x^2 + 5x + k\;(\,x + 1 \\ \pm\,x^2 \pm 4x \\ \hline x + k \\ \pm\,x \pm 4 \Rightarrow\; k - 4 \\ \hline \end{array}$$

For exact division remainder must be zero, k 4 = 0 ⇒ k = 4, so the required value of k is 4.

Thus, $\dfrac{x^2 + 5x + 4}{x + 4} = x + 1$

Here:
$x^2 + 5x + 4$ is dividend
$x + 4$ is divisor
$x + 1$ is quotient and
0 is remainder.

Example 3. Divide $5x^2 - 16xy + 3y^2$ by $x - 3y$ by vertical method

Solution:

$$
\begin{array}{r|l}
x - 3y & 5x - y \\
\hline
5x^2 - 16xy + 3y^2 & \\
5x^2 - 15xy & \\
- \quad + & \\
\hline
-xy + 3y^2 & \\
-xy + 3y^2 & \\
+ \quad - & \\
\hline
\text{remainder} = 0 &
\end{array}
$$

So, the required quotient is $5x - y$

Note: If the remainder is zero then the given polynomial is exactly divisible by the other polynomial.

Explanation:

Step1: Divide 1st. term of the dividend by 1st term of the divisor: $\frac{5x^2}{x} = 5x$

Step 2: Multiply the result with divisor: $5x(x - 3y)$
$= 5x(x - 3y) = 5x^2 - 15xy.$

Step 3: Subtract the result from the dividend and find the remainder.

Step 4: Repeat the process till the exponent of the dividend is less than that of the divisor.

EXERCISE 5.2

1. Add:

(i) $4x + 6y + 5z$, $-3x - 9y$ and $x + 3y - 4z$
(ii) $7x + 2y^3 - 4xy$, $3x - 2xy^3 + 7xy$ and $2xy - 5x + 6y^3$
(iii) $4x^2 + 3y^2 - 6x + 4y - 3$, $2x - y^2 + 3x^2 - 4y + 3$ and $-6x^2 - 2y^2 - 1$
(iv) $6ab - a^2 - b^2 - 7$, $5a^2 - 7ab + 3b^2 + 9$ and $-4b^2 - 3ab - 2a^2 - 3$

2. Subtract:

(i) $2x - 3y - z$ from $z - 4x - 6y$
(ii) $y^4 - 6y^2 - 3y^3 + 4y^5$ from $6y^2 - 3y^5 + 2y^4 + 3y^3 + 8$
(iii) $7x - 8y + 4z - 5w$ from $x^2 + y^2 + z^2 - 8x + 7y - 5z - 5w$
(iv) $5z^2 - 3yz + 2y^2$ from $10y^2 - 2yz - 3z^2$

3. Subtract the sum of $p^2 + 3pq + q^2$ and $2p^2 - pq + 5q^2$ from the sum of $6p^2 - 7pq + 4q^2$ and $7p^2 - 2pq + 3q^2$.

4. The sum of two polynomials is $3a^3 + 3a + 7b + 4ab$. If one polynomial is $4ab - 3a^3$, find the other.

5. Simplify:

(i) $6(2x+y-7xy)-3(5x-2y+5xy)$ (ii) $4(2x-3y+xy)-5(3x-2y-xy)$

(iii) $x(x^2+2xy+y^2)+4y(x^2+3xy+9y^2)$ (iv) $x^2(x^2+xy+y^2)-y^2(x^2-xy-y^2)$

6. Find the product:

(i) $(\sqrt{x}+\sqrt{y})(x-\sqrt{xy}+y)$ (ii) $(x^3-xy+y^3)(x^3+xy+y^3)$

(iii) $(a^2-b^2)(a^2-2ab+b^2)$ (iv) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

7. Perform the division:

(i) $(3x^3-6x^2+12x)\div 3x$ (ii) $(4x^4-16x^3+8x^2)\div 2x^2$

(iii) $(x^2+x-6)\div(x+3)$ (iv) $(9x^2-6xy-8y^2)\div(3x+2y)$

(v) $(a^3+b^3)\div(a+b)$ (vi) $(a^6-b^6)\div(a^2-b^2)$

8. The product of two polynomials is $x^4-3x^2y^2+y^4$. If one polynomial is x^2-xy-y^2, find the other.

9. What should be subtracted from $2a^4+3a^3-a^2-5$, so that it becomes exactly divisible by $a-2$?

10. Find the value of t for which the polynomial $2x^4+3x^3-4x^2+14x+2t$ is exactly divisible by x^2-2x+3.

REVIEW EXERCISE 5

1. Answer the following questions:

(i) What is meant by a cubic polynomial? Write example in one variable.

(ii) Define variable with examples.

(iii) Which literals are used in polynomial $a+2b+3c$?

(iv) How many terms are there in $x^3+2x^2+3x^2$?

(v) What is the degree of a polynomial? Give examples.

2. Tick the correct answer.

(i) The number of variables in ax^2+bx+c is _____.

(a) Zero (b) Three (c) Two (d) One

(ii) $x-1$ is a _____ polynomial.
(a) Quadratic (b) Biquadratic (c) Linear (d) Cubic

(iii) Identify the Quadratic polynomial.
(a) xy (b) $2a^4$ (c) $x+y$ (d) $a+x+3$

(iv) The degree of $3x^2+4xy^4+y^6+9$ is _____.
(a) 9 (b) 4 (c) 5 (d) 6

(V) Number multiplied by a variable or variables in a polynomial is called a _____.
(a) Variable (b) Coefficient (c) Exponent (d) None of these

(vi) _____ is a the cubic polynomial.
(a) $xbc+a^2$ (b) $a^2+b^2+c^2$ (c) $x^3+\frac{1}{x^3}$ (d) 3

(vii) Which algebraic expression is not a polynomial?
(a) $x+\frac{1}{x^{-2}}$ (b) -7 (c) x^3-y^3 (d) $x^{-3}+1$

(viii) The biquadratic polynomial has degree _____.
(a) -1 (b) 3 (c) 4 (d) 0

(ix) In the polynomial px^2+qx+r, literals are:
(a) p, q, r (b) x (c) x^2, x (d) p, q, r, x

(x) If one polynomial is exactly divisible by another polynomial, then the remainder is _____.
(a) 0 (b) 1 (c) -1 (d) None of these

3. Find the value of the following:

(i) $x-5y$ when $x=2a+3b$, $y=a-5b$

(ii) $(x+y)(x^3-2x^2y+xy^2)$ when $x=-2$, $y=-3$

(iii) $\frac{a^2+b^2}{a^2+2ab+b^2}$ when $a=0$, $b=5$

(iv) $\left(a+\frac{1}{a}\right)\left(a^2+1+\frac{1}{a^2}\right)$ when $a=3$

(v) $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ when $a=-2, b=0, c=2$

4. Simplify:

(i) $\frac{x+y}{3}-\frac{y+z}{2}+\frac{4x-z}{5}$ (ii) $(\sqrt{x}+y)(\sqrt{x}-y)$

(iii) $(x+y+2)(x-y-2)$ (iv) $(a^3-2a+4)\div(a+2)$

5. For what value of k, the polynomial k^3-5k^2-14 is divisible by $k+2$?

SUMMARY

- Polynomials are expressions of one or more than one terms whose variables have non-negative integers as their exponents.
- A symbol having a fixed value is called constant.
- A collection of constants and variables connected by fundamental operations (+, –, ×, ÷), extraction of roots and taking powers is called an algebraic expression.
- Variable is a symbol or letter used to show the unknowns, usually we write x, y, z to represent variables. Its value changes.
- A letter used to represent a constant or a variable in an algebraic expression is known as literal. For example in $4a+2b-3c$, a, b and c are literals.
- The degree of a polynomial is the highest degree of its non-zero term.
- The constant appearing as a multiplier of a variable used in an algebraic expression is called its coefficient.
- Coefficients of a polynomial are the Coefficients factors of all variables in its terms.
- Linear polynomials have degree one.
- Quadratic polynomials have degree two.
- Cubic polynomials have degree three.
- Biquadratic polynomials have degree four.
- Every constant has degree zero.
- If all the coefficients in a polynomial are zero, then it is equal to zero with no particular degree.

UNIT 6

FACTORIZATION, SIMULTANEOUS EQUATIONS

6.1 BASIC ALGEBRAIC FORMULAS

6. 1.1 Recall of the formulas:

(i) $(a+b)^2 = a^2 + 2ab + b^2$ **(ii)** $(a-b)^2 = a^2 - 2ab + b^2$

(iii) $a^2 - b^2 = (a-b)(a+b)$

We have already studied and verified the above mentioned formulas in the previous class.

Let us apply them to solve some problems:

Example 1. Evaluate: (i) $(102)^2$ (ii) $(98)^2$ (iii) (102) (98)

Solution:

(i) $(102)^2 = (100+2)^2$

$= (100)^2 + 2(100)(2) + (2)^2$

[using $(a+b)^2 = a^2 + 2ab + b^2$]

$= 10000 + 400 + 4 = 10404$

(ii) $(98)^2 = (100-2)^2$

$= (100)^2 - 2(100)(2) + (2)^2$,

[using $(a-b)^2 = a^2 - 2ab + b^2$]

$= 10000 - 400 + 4 = 9604$

(iii) (102) (98) = (100 + 2) (100 – 2)

$= (100)^2 - (2)^2$,

Using: $[(a+b)(a-b) = a^2 - b^2]$

$= 10000 - 4 = 9996$

Activity: Solve 203 × 197

(200 + ——) × (—— – 3)

using $(a+b)(a-b) = (\quad)$

$= (200)^2 - (\quad)^2$ ————

= 40,000 – ———— = ————

Example 2. Evaluate: (i) $(1.02)^2$ (ii) $(0.98)^2$ (iii) (1.02) (0.98)

Solution:

(i) $(1.02)^2 = (1 + .02)^2$

$= (1)^2 + 2(1)(0.02) + (0.02)^2$,

[**Using:** $\because (a+b)^2 = a^2 + 2ab + b^2$]

$= 1 + 0.04 + 0.0004 = 1.0404$

(ii) $(0.98)^2 = (1 - 0.02)^2$

$= (1)^2 - 2(1)(0.02) + (0.02)^2$,

[using $(a-b)^2 = a^2 - 2ab + b^2$]

$= 1 - 0.04 + 0.0004 = 0.9604$

(iii) (1.02) (0.98) = (1 + 0.02) (1 – 0.02) = $(1)^2 - (0.02)^2$,

[**Using:** $(a+b)(a-b) = a^2 - b^2$]

$= 1 - 0.0004 = 0.9996$

Example 3. Find the value $x^2+\frac{1}{x^2}$ and $x^4+\frac{1}{x^4}$ when (i) $x+\frac{1}{x}=6$ (ii) $x-\frac{1}{x}=2$

Solution:

(i) $x+\frac{1}{x}=6$... Given

$\Rightarrow (x+\frac{1}{x})^2=(6)^2$ (Squaring both sides)

$\Rightarrow x^2+2(x)(\frac{1}{x^2})+\frac{1}{x^2}=36$

$\Rightarrow x^2+2+\frac{1}{x^2}=36$

$\Rightarrow x^2+\frac{1}{x^2}=36-2$

$\Rightarrow x^2+\frac{1}{x^2}=34$

$\Rightarrow (x^2+\frac{1}{x^2})^2=(34)^2$ (Squaring both sides)

$\Rightarrow x^4+2(x^2)(\frac{1}{x^2})+\frac{1}{x^4}=1156$

$\Rightarrow x^4+\frac{1}{x^4}=1156-2=1154$

(ii) $x-\frac{1}{x}=2$... given

$\Rightarrow (x-\frac{1}{x})^2=(2)^2$ (Squaring both sides)

$\Rightarrow x^2-2(x)(\frac{1}{x})+\frac{1}{x^2}=4$

$\Rightarrow x^2-2+\frac{1}{x^2}=4$

$\Rightarrow x^2+\frac{1}{x^2}=4+2$

$\Rightarrow x^2+\frac{1}{x^2}=6$

$\Rightarrow (x^2+\frac{1}{x^2})^2=(6)^2$ (Squaring both sides)

$\Rightarrow x^4+2(x^2)(\frac{1}{x^2})+\frac{1}{x^4}=36$

$\Rightarrow x^4+\frac{1}{x^4}=36-2=34$

Example 4.

Show that $x^4+\frac{1}{x^4}=(x^2+\frac{1}{x^2})^2-2$

Proof: R.H.S $=(x^2+\frac{1}{x^2})^2-2$

$=(x^2)^2+2(x^2)(\frac{1}{x^2})+(\frac{1}{x^2})^2-2$

$=x^4+2+\frac{1}{x^4}-2=x^4+\frac{1}{x^4}$ = L.H.S. Hence proved.

EXERCISE 6.1

1. Evaluate the following by using suitable formula:

(i) $(105)^2$ (ii) $(96)^2$ (iii) $(57)^2$ (iv) $(52)^2$

(v) 104×96 (vi) 47×53 (vii) 107×93 (viii) 89×111

2. Evaluate the following by using suitable formula:

(i) $(1.03)^2$ (ii) $(0.99)^2$ (iii) $(1.05)^2$
(iv) $(0.91)^2$ (v) 1.03×0.97 (vi) 5.02×4.98

3. Find the value of $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$ when:

(i) $x + \frac{1}{x} = 5$ (ii) $x + \frac{1}{x} = -4$ (iii) $x + \frac{1}{x} = 7$

4. Find the value of $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$ when:

(i) $x - \frac{1}{x} = 3$ (ii) $x - \frac{1}{x} = 0.2$ (iii) $x - \frac{1}{x} = -6$

5. Show that: (i) $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2$ (ii) $x^2 + \frac{1}{x^2} = (x - \frac{1}{x})^2 + 2$

6.2 FACTORIZATION

If the product of two or more expressions is equal to the given expression, then these two or more expressions are called the factors of the given expression.

For example; if $3x^2 + 6x = 3x(x + 2)$, then 3, x and $(x + 2)$ are the factors of $3x^2 + 6x$.

The process of finding factors of a given expression is called **factorization**. We express the given expression as a product of two or more expressions.

6.2.1 Factors of the expression of the type: $ka + kb + kc$

We know that if k is a non-zero real number then

$$ka + kb + kc = k(a + b + c)$$

Here k and $(a + b + c)$ are the factors of $k(a + b + c)$

Example 1. Factorize: (i) $5x + 10y + 20z$ (ii) $6x^2 + 12xy - 30xy^2$

Solution: (i) $5x + 10y + 20z$
$= 5(x + 2y + 4z)$,
($\because$ 5 is a common factor)

(ii) $6x^2 + 12xy - 30xy^2$
$= 6x(x + 2y - 5y^2)$,
($\because$ $6x$ is a common factor)

6.2.2 Factors of the expression of the type: $ac + ad + bc + bd$

We show that: $ac + ad + bc + bd = (a + b)(c + d)$

Proof: L.H.S. $= ac + ad + bc + bd = a(c + d) + b(c + d)$

$= (a + b)(c + d) =$ R.H.S.

Example 2. Factorize: (i) $5x + xz + 5z + z^2$ (ii) $3x^2y + 6xy^2 - 2xz - 4yz$

Solution: **(i)** $5x + xz + 5z + z^2$

$= x(5 + z) + z(5 + z)$

$= (5 + z)(x + z)$

(ii) $3x^2y + 6xy^2 - 2xz - 4yz$

$= 3xy(x + 2y) - 2z(x + 2y)$

$= (x + 2y)(3xy - 2z)$

6.2.3 Factors of the expression of the type: $a^2 \pm 2ab + b^2$

(i) We show that: $a^2 + 2ab + b^2 = (a + b)^2$

Proof: L.H.S. $= a^2 + 2ab + b^2$

$= a^2 + ab + ab + b^2$

$= a(a + b) + b(a + b)$

$= (a + b)(a + b)$

$= (a + b)^2 =$ R.H.S

(ii) We also show that: $a^2 - 2ab + b^2 = (a - b)^2$

Proof: L.H.S.

$= a^2 - 2ab + b^2$

$= a^2 - ab - ab + b^2$

$= a(a - b) - b(a - b)$

$= (a - b)(a - b) = (a - b)^2 =$ R.H.S.

Example 1. Factorize: $x^2 + 6xy + 9y^2$

Solution: $x^2 + 6xy + 9y^2$

$= (x)^2 + 2(x)(3y) + (3y)^2$

$= (x + 3y)^2$, (using formula)

Example 2. Factorize: $25x^2 - 10xy + y^2$

Solution: $25x^2 - 10xy + y^2$

$= (5x)^2 - 2(5x)(y) + (y)^2 = (5x - y)^2$

6.2.4 Factors of the expression of the type: $a^2 - b^2$

We show that: $a^2 - b^2 = (a + b)(a - b)$

Proof: R.H.S $= (a + b)(a - b)$

$= a(a - b) + b(a - b)$

$= a^2 - ab + ab - b^2$

$= a^2 - b^2 =$ L.H.S

Example 1.

Factorize: $4a^2 - 9b^2$

Solution: $4a^2 - 9b^2$

$= (2a)^2 - (3b)^2$

$= (2a + 3b)(2a - 3b)$

Example 2. Factorize: **(i)** $25x^2 - 36y^2$ **(ii)** $8a^2 - 50b^2$

Solution: **(i)** $25x^2 - 36y^2$

$= (5x)^2 - (6y)^2$

$= (5x + 6y)(5x - 6y)$

(ii) $8a^2 - 50b^2$

$= 2(4a^2 - 25b^2)$

$= 2\{(2a)^2 - (5b)^2\}$

$= 2(2a + 5b)(2a - 5b)$

6.2.5 Factors of the expression of the type: $a^2 \pm 2ab + b^2 - c^2$

In this type of expressions, we first apply the formula $x^2 + 2xy + y^2 = (x + y)^2$ and then $a^2 - b^2 = (a + b)(a - b)$

Example 1. Factorize $x^2 + 2(x)(5) + 25 - y^2$

Solution: $x^2 + 10x + 25 - y^2 = [(x)^2 + (x)(10) + (5)^2] - y^2$

$= (x + 5)^2 - y^2 = (x + 5 + y)(x + 5 - y)$

Similarly we factorize the expression of the type $a^2 - 2ab + b^2 - c^2$ by using the formulas $a^2 - 2ab + b^2 = (a - b)^2$ and $x^2 - y^2 = (x + y)(x - y)$

Example 2. Find factors of $x^2 - 2xy + y^2 - z^2$

Solution: $x^2 - 2xy + y^2 - z^2 = [(x)^2 - 2(x)(y) + (y)^2] - z^2$

$= (x - y)^2 - z^2 = (x - y + z)(x - y - z)$

EXERCISE 6.2

A. Factorize the following:

(1) $4x + 8z$

(2) $5x + 10y + 30z$

(3) $2x - 4xy + 8xz$

(4) $2a^2 + 10a^3 - 20a^4$

(5) $3a^2b + 7ab^2 - 8a^2b^2$

(6) $6a^2bc + 12ab^2c - 36abc^2$

(7) $5x + 10y + 3xz + 6yz$

(8) $abc - abd + cx - xd$

(9) $x^2 + 5x + 6xy + 30y$

(10) $7xy + 14yz - 5ax - 10az$

(11) $3x^2 + 6y^2 + 6x^2z + 12y^2z$

(12) $x^2 - 7xy - xz + 7yz$

B. Resolve into factors:

(1) $a^2 + 10a + 25$

(2) $x^2 + 12xy + 36y^2$

(3) $4x^2 + 12xy + 9y^2$ **(4)** $16a^2 + 40ab + 25b^2$

(5) $b^2 + c^2 - 2bc$ **(6)** $49p^2 - 14p + 1$

(7) $81c^2 - 36cd + 4d^2$ **(8)** $144x^4 - 72x^2y^2 + 9y^2$

(9) $3a^2 - 6ab + 3b^2$ **(10)** $x^2 - 3xy + y^2$

C. Find factors of the following:

(1) $81x^2 - 4y^2$ **(2)** $169a^2 - 100b^2$ **(3)** $3a^2 - 27b^4$

(4) $2p^2 - 18q^2$ **(5)** $5x^2 - 125y^2$ **(6)** $\frac{25}{x^2} - \frac{y^2}{16}$

(7) $\frac{x^2}{144} - y^2$ **(8)** $\frac{36}{25} l^2 - \frac{49}{4} d^2$ **(9)** $c^2 - (a-b)^2$

(10) $(x+y)^2 - z^2$ **(11)** $(a+b)^2 - (p+q)^2$ **(12)** $144 - y^4$

(13) $32a^2 - 50b^2$ **(14)** $x^4 - y^4$ **(15)** $(2a+b) - 9c^2$

D. Resolve into factors:

(1) $x^2 + 2xy + y^2 - z^2$ **(2)** $a^2 + b^2 - 2ab - c^2$

(3) $64a^2 + 48ab + 9b^2 - c^2$ **(4)** $4p^2 - 12pq + 9q^2 - 49r^2$

(5) $x^4 + 2x^2y^2 + y^4 - 25z^2$ **(6)** $\frac{x^2}{4} - xy + y^2 - \frac{c^2}{36}$

(7) $2a^2 + 4ab + 2b^2 - 2c^2$ **(8)** $81c^2 - 16p^2 + 18cd + d^2$

(9) $4x^2 + 12x + 9 - 16y^2$ **(10)** $36 - 25a^2 + 70ab - 49b^2$

6.3 MANIPULATION OF ALGEBRAIC EXPRESSION

Now we recognize and apply following formulas to solve problems.

6.3.1 Formula for the cube of the sum of two terms

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Proof: L.H.S. $= (a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2 + 2ab + b^2)$

$= a(a^2 + 2ab + b^2) + b(a^2 + 2ab + b^2)$

$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3 =$ R.H.S.

Note: $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

$\Rightarrow \quad a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

Example 1. Find the cube of $(2x + 3)$

Solution: $(2x+3)^3 = (2x)^3 + 3(2x)^2(3) + 3(2x)(3)^2 + (3)^3$

$= 8x^3 + 3(4x^2)(3) + 6x(9) + 27$

$= 8x^3 + 36x^2 + 54x + 27$

Example 2. Find the value of $x^3 + \frac{1}{x^3}$ when $x + \frac{1}{x} = 5$

Solution: We have $x + \frac{1}{x} = 5$

Method I: From known to unknown.

Cubing both sides, by formula

$(x + \frac{1}{x})^3 = (5)^3$

$x^3 + 3(x^2)(\frac{1}{x}) + 3(x)(\frac{1}{x^2}) + \frac{1}{x^3} = 125$

$x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = 125$

$\Rightarrow x^3 + 3(x + \frac{1}{x}) + \frac{1}{x^3} = 125$

$x^3 + 3(5) + \frac{1}{x^3} = 125 \quad (\because x + \frac{1}{x} = 5)$

$\Rightarrow x^3 + \frac{1}{x^3} = 125 - 15 = 110.$

Method II:
From unknown to known.

We know that:

$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$\Rightarrow x^3 + \frac{1}{x^3} = (x + \frac{1}{x})^3 - 3(x)(\frac{1}{x})(x + \frac{1}{x})$

$\Rightarrow x^3 + \frac{1}{x^3} = (x + \frac{1}{x})^3 - 3(x + \frac{1}{x})$

$= 5^3 - 3(5)$

$= 125 - 15$

$= 110.$

6.3.2 Formula for the cube of the difference of two terms

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

Proof: L.H.S. $= (a-b)^3 = (a-b)(a-b)^2 = (a-b)(a^2 - 2ab + b^2)$

$= a(a^2 - 2ab + b^2) - b(a^2 - 2ab + b^2) = a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$

$= a^3 - 3a^2b + 3ab^2 - b^3 =$ R.H.S.

Example 1. **Find the cube of $(4x-1)$**

Solution: $(4x-1)^3$

$= (4x)^3 - 3(4x)^2(1) + 3(4x)(1)^2 - (1)^3$

$= 64x^3 - 3(16x^2) + 12x - 1$

$= 64x^3 - 48x^2 + 12x - 1$

Example 2. **Find the cube of 19 using the formula.**

Solution: $(19)^3 = (20-1)^3$

$= (20)^3 - 3(20)^2(1) + 3(20)(1)^2 - (1)^3$

$= 8000 - 1200 + 60 - 1 = 6,859$

6.3.3 Find $x^3 \pm \frac{1}{x^3}$ when the value of $x \pm \frac{1}{x}$ is given

Example 3. Find the value of $x^3 - \frac{1}{x^3}$ when $x - \frac{1}{x} = 2$

Solution: $x - \frac{1}{x} = 2$

Method I: Known to unknown.

$(x - \frac{1}{x})^3 = (2)^3$ Cubing both sides,

$\Rightarrow x^3 - 3(x^2)(\frac{1}{x}) + 3(x)(\frac{1}{x^2}) - \frac{1}{x^3} = 8,$

$\Rightarrow x^3 - 3x + \frac{3}{x} - \frac{1}{x^3} = 8$

$\Rightarrow x^3 - 3(x - \frac{1}{x}) - \frac{1}{x^3} = 8$

$\Rightarrow x^3 - 3(2) - \frac{1}{x^3} = 8 \quad (\because x - \frac{1}{x} = 2)$

$\Rightarrow x^3 - 6 - \frac{1}{x^3} = 8$ or $x^3 - \frac{1}{x^3} = 8 + 6 = 14$

Method II:

Unknown in terms of known.

$a^3 - b^3 = (a-b)^3 + 3ab(a-b)$

$x^3 - \frac{1}{x^3} = (x - \frac{1}{x})^3 + 3x\frac{1}{x}(x - \frac{1}{x})$

$= (x - \frac{1}{x})^3 + 3(x - \frac{1}{x})$

$= (2)^3 + 3(2) = 8 + 6 = 14$

EXERCISE 6.3

1. Find the cube of each of the following:

(i) $2x + 3$ (ii) $5x - 1$ (iii) $2x - 3y$ (iv) $2a + 5b$ (v) $x + 7z$

(vi) $\frac{x}{2} - 1$ (vii) $5 + 3a$ (viii) $5 - \frac{a}{4}$ (ix) $5a - 3b$ (x) $\frac{x}{3} + \frac{y}{5}$

2. Find the cube of each of the following by using the formula:

(i) 18 (ii) 13 (iii) 105 (iv) 10.1 (v) 0.98 (vi) 2.25

3. Find the value of $x^3 + \frac{1}{x^3}$ when:

(i) $x + \frac{1}{x} = 6$ (ii) $x + \frac{1}{x} = -5$

(iii) $x + \frac{1}{x} = \frac{1}{2}$ (iv) $x + \frac{1}{x} = 10.1$

4. Find the value of $x^3 - \frac{1}{x^3}$ when:

(i) $x - \frac{1}{x} = 5$ (ii) $x - \frac{1}{x} = -2$

(iii) $x - \frac{1}{x} = \frac{1}{3}$ (iv) $x - \frac{1}{x} = 9.9$

6.4 SIMULTANEOUS LINEAR EQUATIONS

We know that a linear equation is an equation with degree one. It may be in one, two or more variables.

For example, $2x + 6 = 0$ is a linear equation in one variable x. Similarly $5x + 6y = 9$ is a linear equation in two variables x and y.

Standard form of a linear equation in one variable is: $ax + b = 0$, where a, b are real numbers and $a \neq 0$. For example, $2x + 3 = 0$.

Standard form of a linear equation in two variables is: $ax + by + c = 0$, where a, b, c are real numbers with $a \neq 0$ and $b \neq 0$.

6.4.1 Recognize Simultaneous Linear Equations in one and two variables

Two or more linear equations which occur in one and the same problem are called simultaneous linear equations.

Example 1. $2x + 3 = 5$ and $2x + y = 6$ are two simultaneous linear equations in one and two variables respectively.

Example 2. $6x + y = 8$ and $3x - y = 1$ are two simultaneous linear equations in two variables.

Note: Simultaneous linear equations have only one solution. As in example 1, the solution is $x = 1$, $y = 4$ or $(x, y) = (1, 4)$. In example 2 the solution is $x = 1$, $y = 2$ or $(x, y) = (1, 2)$.

6.4.2 Give the concept of formation of a linear equation in two variables.

We know that a linear equation in two variables is in fact an algebraic representation of a problem or statement involving two quantities.

In order to form a linear equation we have to follow following steps.

Step 1. Read the problem or statement carefully and identify the unknown quantities $(x, y, z, t \ldots)$

Step 2. Represent the unknown quantities with letters of the English alphabet.

Step 3. Write the equation which satisfies the condition of given problem or statement.

Example 1. Form a linear equation for each of the following statements.

(i) The sum of ages of two students is 30 years.

(ii) Total amount spent for 5 kilogram of apples and 3 kilograms of mangoes is 550 rupees.

(iii) Price of 6 pens is double the price of a book.

Solution: (i) Let the ages of two students in years be x and y. So, the required equation is: $x + y = 30$.

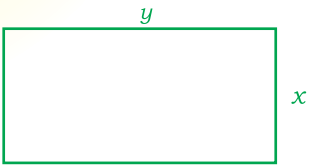
(ii) Let the price of 1kg of apples be x rupees and the price of 1kg of mangoes be y rupees. The required equation will be $5x + 3y = 550$

(iii) Let the price of a pen be x rupees and that of a book be y rupees. So, the required equation will be $6x = 2y$ or $6x - 2y = 0$

Example 2. Form two simultaneous linear equations in two variables for the statement.

The length of a rectangle is greater than its width by 4 cm and perimeter of the rectangle is 192 cm.

Solution: Let length = y and width = x



Then $y - x = 4$ … (i)

Now Perimeter = $2(x + y) = 192$ cm

or $x + y = 96$ … (ii)

6.4.3 Important facts about linear equations know that:

(1) A single linear equation in two unknowns is satisfied by as many pair of values as required.

For example, $x + y = 6$ is a single linear equation in two variables and its solutions are infinite. Some of them are (1, 5), (2, 4), (3, 3), (4, 2) etc.

Verification: We have $x + y = 6$,
for (1, 5), we get $1 + 5 = 6$ (satisfied) for (2, 4), we get $2 + 4 = 6$ (satisfied)
for (3, 3), we get $3 + 3 = 6$ (satisfied) for (4, 2), we get $4 + 2 = 6$ (satisfied).

(2) Two simultaneous linear equations in two unknowns have only one solution (i.e., one pair of values)

Either both of these are in two variables or one of them is in one variable.

Note: Solution of two simultaneous equations means the values of the unknowns which satisfy both the equations.

For example, $x + y = 5$ and $x - y = 3$ are two simultaneous linear equations and they have only one solution, i.e., (4, 1).

Verification: We have $x + y = 5$ and $x - y = 3$,
for (4, 1), we get $4 + 1 = 5$ and $4 - 1 = 3$
so both the equations are satisfied by (4, 1).

EXERCISE 6.4

1. Identify and separate linear equations in one variable, linear equations in two variables and simultaneous linear equations.

(i) $2x + 3y = 6$ (ii) $2x - 7 = 0$ (iii) $2x = 6$, $x + y = 8$
(iv) $x - y = 7$, $x + y = 10$ (v) $5y = 8$ (vi) $3y - 7 = 8x$

2. Form a linear equation for each of the following statements:

(i) Sum of weights of two boys is 50kg.
(ii) The price of six pens is three times the price of a magazine.
(iii) If twice of a number is added to 5, the result is 25.
(iv) If the price of 6kg of mangoes is subtracted from the price of 3kg of guavas, the result is 45 rupees.
(v) Number of girls in a school is two-third of the number of boys.

3. Guess and verify any three solutions of: (i) $x + y = 10$ (ii) $2x + y = 10$

4. Guess and verify any four solutions of: (i) $x - y = 2$ (ii) $2x + y = 16$

5. Guess and verify the solution of $x - y = 4$ and $x - 2y = 2$

6. Which one of the pair (1, 5) and (6, 0) is not a solution of the simultaneous equations $2x + y = 7$ and $6x - y = 1$?

7. Form a pair of simultaneous linear equations for the following statements:

(i) The price of a book is 50 rupees more than a notebook and their total price is Rs 115.
(ii) The sum of two numbers is 27 and difference is 17.
(iii) Price of a pen is Rs 50 and the price of 2 pens and 5 pencils is Rs 200.
(iv) The measure of an angle is double that of its complementary angle.

6. 5 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

There are several methods to solve two simultaneous linear equations, here we will discuss only three methods.

6.5.1 Solve Simultaneous linear equations using

(1) Method of equating the coefficients

We have already learnt the solution of a single linear equation in previous class.

In this method, we make coefficients of a variable equal if they are different and then add or subtract the resultant equations to eliminate that variable as is explained in the following example.

Example 1. Solve $2x - 3y = 3$ and $5x + y = 16$ by using the method of equating the coefficients.

Solution: We have: $2x - 3y = 3$ .... (i) $\quad 5x + y = 16$ .... (ii)

In order to make co-efficients of x equal. we get:

Multiply eq: (i) by 5, Multiply eq: (ii) by 2

$10x - 15y = 15$ .... (iii)

$10x + 2y = 32$ .... (iv)

Subtracting eq: (iv) from eq: (iii)

$10x - 15y = 15$

$\pm 10x \pm 2y = \pm 32$

$-17y = -17$ (x is eliminated)

$\Rightarrow \quad 17y = 17$ or $y = \frac{17}{17} = 1.$

By putting $y = 1$ in eq: (i),

we get $2x - 3(1) = 3$

$\Rightarrow \quad 2x - 3 = 3$

$\Rightarrow \quad 2x = 6$

$\Rightarrow \quad x = \frac{6}{6} = 3$

So, $x = 3$ and $y = 1$

Thus, the solution set = {(3, 1)}.

Note 1: If coefficient of any variable in one of the equations is 1, then multiply only that equation.

Note 2: Try to multiply by smaller numbers.

(2) Method of elimination by substitution

In this method, we find the value of one variable in terms of the other from any one of the equations and use that value in the other equation to eliminate one variable as explained in the following example.

Example 2. Solve: $x + 2y = 8$ and $3x - 4y = -6$ by using the method Of elimination by substitution.

Solution: We have

$x + 2y = 8$ .... (i)

$3x - 4y = -6$ .... (ii)

From eq: (i), we get: $x = 8 - 2y$ .... (iii)

By putting $x = 8 - 2y$ in eq: ..... (ii),

we get $3(8 - 2y) - 4y = -6$

$\Rightarrow \quad 24 - 6y - 4y = -6$

$\Rightarrow \quad 24 - 10y = -6$

$\Rightarrow \quad -10y = -6 - 24$

or $\quad -10y = -30$

or $\quad 10y = 30$

$\Rightarrow \quad y = \frac{30}{10} = 3.$

By using $y = 3$ in eq: (iii), we get $x = 8 - 2(3) = 8 - 6 = 2$
So, $x = 2$ and $y = 3$. Thus, the solution set $= \{(2, 3)\}$

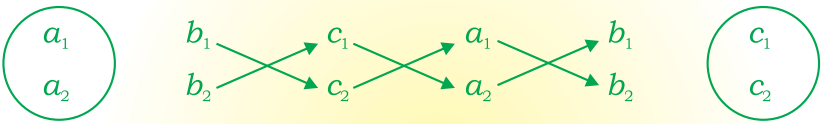
(3) Method of cross-multiplication

In order to explain this method, we take two simultaneous equations in standard form, i.e., $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Following are the steps of this method.

Step 1: We write coefficients of variables and constants in the following manner:

$$\begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 & c_2 \end{matrix}$$

Step 2: We cross multiply in the following way after excluding first and last column.



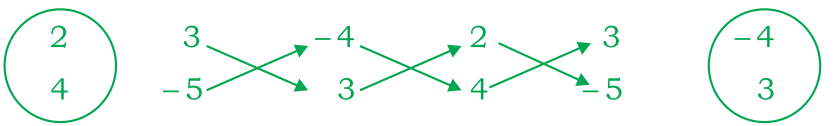
Step 3: In order to find values of x and y, we use the following equal ratios.

$$\frac{x}{b_1\,c_2 - b_2\,c_1} = \frac{y}{c_1\,a_2 - c_2\,a_1} = \frac{1}{a_1\,b_2 - a_2\,b_1}$$

Example 3. Solve: $2x + 3y = 4$ and $4x - 5y = -3$ by using method of cross-multiplication.

Solution: We have: $2x + 3y - 4 = 0$ and $4x - 5y + 3 = 0$. (Standard form)

Writing the coefficients of the variables and excluding the first and last column or arranging them in the following manner and cross-multiplying, we get



By cross-multiplication method,

$$\frac{x}{(3)\,(3) - (-5)\,(-4)} = \frac{y}{(-4)\,(4) - (3)\,(2)} = \frac{1}{(2)\,(-5) - (4)\,(3)}$$

$\Rightarrow \quad \frac{x}{9-20} = \frac{y}{-16-6} = \frac{1}{-10-12}$ or, $\frac{x}{-11} = \frac{y}{-22} = \frac{1}{-22}$

Two equations: $\frac{x}{-11} = \frac{1}{-22}$ and $\frac{y}{-22} = \frac{1}{-22}$

$\Rightarrow \quad -22x = -11$ and $y = \frac{22}{22} = 1$

So, $x = \frac{1}{2}$ and $y = 1$.

Thus, the solution set $= \left\{\left(\frac{1}{2}, 1\right)\right\}$ or $\{(0.5, 1)\}$.

EXERCISE 6.5

1. Solve the following simultaneous equations by using the method of equating the coefficients.

(i) $2x + y = 10$, $5x - 2y = 7$

(ii) $5x - y = 3$, $3x + y = 4$

(iii) $3x + 7y = 27$, $5x + 2y = 16$

(iv) $2x + 3y = 2$, $4x - 9y = -1$

(v) $2x + y = 8$, $6x - 2y = -1$

(vi) $2x + 3y = 14$, $3x + 2y = 5$

2. Solve the following simultaneous equations by using the method of elimination by substitution.

(i) $3x - 5y = 5$, $2x + y = 12$

(ii) $4x + 3y = 2$, $5x + 2y = -1$

(iii) $13x - 9y = 1$, $11x - 12y = 14$

(iv) $7x - 4y = 1$, $2x - 5y = 8$

(v) $x + y = 1$, $3x + 6y = 5$

(vi) $3x - 7y = 1$, $4x + y = 22$

3. Solve the following simultaneous equations by the method of cross-multiplication.

(i) $2x + 5y = 7$, $5x - y = 4$

(ii) $-2x + 5y = 5$, $3x - 2y = -13$

(iii) $2x + y = 10$, $x + 3y = 8$

(iv) $x - 4y = 2$, $2x - 7y = 5$

(v) $10x + 2y = 8$, $5x + y = 4$

(vi) $13x - y = 14$, $2x - 7y = 9$

6.5.2 Solve real life problems involving two simultaneous linear equations in two variables

Let us solve some real life problems involving two simultaneous equations in two variables.

Example 1. Aslam spent Rs 400 to purchase 3 kg of apples and 2 kg of mangoes. Find the price of 1kg of apples if the rate of apples is double the rate of mangoes.

Solution: Let the price of 1 kg of apples be x rupees and the price of 1 kg of mangoes be y rupees.

According to the condition of the problem:

$3x + 2y = 400$ —— (i)

and $x = 2y$ —— (ii)

By putting $x = 2y$ in eq: (i),

we get $3(2y) + 2y = 400$

$\Rightarrow \quad 6y + 2y = 400$

$\Rightarrow \quad 8y = 400$

$\Rightarrow \quad y = \frac{400}{8} = 50$

By putting $y = 50$, in eq. (ii)

we get $x = 2(50) = 100$.

So, the price of 1kg of apples is 100 rupees.

Example 2. Bisma spent 350 rupees to purchase some pens at the rate of Rs 50 per pen and some copies at the rate of Rs 30 per copy. How many pens did she purchase if the total number of pens and copies was 9?

Solution: Le the number of pens be x and that of copies be y.

According to the condition of the problem:

$x + y = 9$ —— (i)

and $50x + 30y = 350$ —— (ii)

From eq: (i), $x = 9 - y$ —— (iii)

By putting $x = 9 - y$ in eq: (ii), we get

$50(9 - y) + 30y = 350$

$\Rightarrow \quad 450 - 50y + 30y = 350$

$\Rightarrow \quad 450 - 20y = 350$

or $\quad -20y = 350 - 450$

or $\quad -20y = -100$

or $\quad 20y = 100$

$\Rightarrow \quad y = \frac{100}{20} = 5.$

Now by putting $y = 5$ in eq: (iii), we get: $x = 9 - 5 = 4$

So, she purchased 4 pens and 5 copies.

EXERCISE 6.6

1. Aisha spent Rs 300 for purchasing 6 kg potatoes. Find the rate of tomatoes per kg if the rate of tomatoes is thrice the rate of potatoes.

2. Taha purchased bats at the rate of Rs 100 per bat and balls at the rate of Rs 50 per ball. How many balls did he purchase if he spent Rs 600 for eight items?

3. The length of a rectangular playground is double that of its width. If the cost of fencing the whole ground is Rs 9000 at the rate of Rs 30 per metre, find the dimensions of the ground.

4. The number of boys in a school is double that of the girls . What is the number of boys, if the total number of students in the school is 450?

5. The ratio of monthly income of a labourer to its monthly expenditure is 15:14. What is his income, if his monthly saving is Rs 2000?

6.6 ELIMINATION

The method of obtaining a relation from two or more equations, free of a particular variable is called **elimination.**

Note: The relation obtained after elimination is called the eliminant.

6.6.1 Eliminate a variable from two equations by: (a) Substitution (b) Application of the formulae.

There are several methods to eliminate a variable from two equations of two unknowns. However, here we will discuss only two methods.

(i) Elimination by Substitution

In this method, we find the value of a variable (which is to be eliminated) in terms of the other variable from an equation and then that value is put in the other equation to eliminate the required variable.

Example 1. Eliminate x from the following equations:

$2x + y = 6$ ..... (i)

and $x - 3y = 8$..... (ii)

Solution: We have

$2x + y = 6$ —— (i)

$x - 3y = 8$ —— (ii)

From eq: (ii), $x = 8 + 3y$

By putting $x = 8 + 3y$ in eq: (i), we get

$2(8 + 3y) + y = 6$

$\Rightarrow 16 + 6y + y = 6$

$\Rightarrow 7y = 6 - 16$

$\Rightarrow 7y + 10 = 0$

or $7y = -10$

Thus, the relation : $7y = -10$ is free of x.

Example 2. Eliminate Vi from the following two equations of motion: $V_f^2 - V_i^2 = 2gS$ and $V_f = V_i + gt$, where V_i and V_f stand for initial and final velocities.

Solution: We have

$V_f^2 - V_i^2 = 2gS \rightarrow$ (i)

$V_f = V_i + gt \rightarrow$ (ii)

From eq: (ii), $V_i = V_f - gt$

By using $V_i = V_f - gt$ in eq: (i), we get $V_f^2 - (V_f - gt)^2 = 2gS$

$\Rightarrow V_f^2 - (V_f^2 - 2gtV_f + g^2t^2) = 2gS$

$V_f^2 - V_f^2 + 2gtV_f - g^2t^2 = 2gS$

$\Rightarrow g(2V_f t - gt^2) = 2gS$

$\Rightarrow 2V_f t - gt^2 = 2S$

$\Rightarrow S = \frac{2V_f t}{2} - \frac{gt^2}{2}$

$\Rightarrow S = V_f t - \frac{1}{2} gt^2$.

Thus,the relation $S = V_f t - \frac{1}{2} gt^2$ is free of Vi.

(ii) Elimination by application of formulas

In this method, we apply formulas such as $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 - 2ab + b^2$ according to the condition of the problem.

Example 3. Eliminate x from $x + \frac{1}{x} = a$ and $x - \frac{1}{x} = b$.

Solution: We have

$x + \frac{1}{x} = a$ —— (i)

$x - \frac{1}{x} = b$ —— (ii)

Squaring both sides of eq: (i)

and eq: (ii), we get

$x^2 + 2 + \frac{1}{x^2} = a^2$ —— (iii)

and $x^2 - 2 + \frac{1}{x^2} = b^2$ —— (iv)

Now subtracting eq: (iv) from eq: (iii), we get

$x^2 + 2 + \frac{1}{x^2} = a^2$

$x^2 - 2 + \frac{1}{x^2} = b^2$

$- \quad + \quad -$

$4 = a^2 - b^2$

Thus, the relation $a^2 - b^2 = 4$ is free of x.

EXERCISE 6.7

1. Eliminate x by substitution method.

(i) $x - y + 1 = 0$, $3x + y - 3$ (ii) $x + 5y = 2$, $6x + 3y = 7$ (iii) $2x + y = 7$, $x + 3y = 4$

(iv) $x^2 + y = 0$, $ax + b = 0$ (v) $x - 2y + 1 = 0$, $3x + y - 3 = 0$ (vi) $4x + 3y + 8 = 0$, $x + 5y - 2 = 0$

2. Eliminate V_i by substitution method.

(i) $V_f = V_i + at$, $S = V_i t + \frac{1}{2} at^2$ (ii) $V_f = V_i + gt$, $S = V_i t - \frac{1}{2} gt^2$ (iii) $V_f^2 - V_i^2 = 2gS$, $V_f = V_i + gt$

3. Eliminate x by using formula.

(i) $x + \frac{1}{x} = 2a$, $x - \frac{1}{x} = 2b + 1$ (ii) $x^2 - \frac{1}{x^2} = 2a$, $x^2 + \frac{1}{x^2} = 3b$ (iii) $\sqrt{x} + \frac{1}{\sqrt{x}} = p$, $x - \frac{1}{x} = q^2$

(iv) $x^2 + \frac{1}{x^2} = p^2$, $x^4 + \frac{1}{x^4} = q^4$ (v) $x + \frac{1}{x} = 2y + 1$, $x - \frac{1}{x} = y - 2$ (vi) $\frac{x^3}{y^3} + \frac{y^3}{x^3} = m$, $\frac{x^3}{y^3} - \frac{y^3}{x^3} = n$

REVIEW EXERCISE 6

1. Choose the correct answer.

(i) $(x + y)^2 =$ _____.

(a) $x^2 + y^2$ (b) $x^2 + 2xy + y^2$ (c) $x^2 - y^2$ (d) both a and b

(ii) $x^2 - y^2 =$ _____.

(a) $(x + y)^2$ (b) $(x - y)^2$ (c) $x^2 + y^2$ (d) $(x - y)(x + y)$

(iii) $y^3 + 3y^2x + 3yx^2 + x^3 =$ _____.

(a) $(y - x)^3$ (b) $(x - y)^3$ (c) $(x + y)^3$ (d) None of these

(iv) A solution of $x + y + 6 = 0$ is_____.

(a) $(1, 2)$ (b) $(-2, 4)$ (c) $(-4, -2)$ (d) $(4, 2)$

(v) Solution of the system: $x+y=5$ and $x-y=1$ is _____.

(a) (2, 3) (b) (3, 2) (c) (1, 4) (d) (4, 1)

2. Find the value of: (i) $x^2+\frac{1}{x^2}$ (ii) $x^4+\frac{1}{x^4}$ (iii) $x^4-\frac{1}{x^4}$ (iv) $x^3+\frac{1}{x^3}$

(v) $x^3-\frac{1}{x^3}$ (vi) $x^6-\frac{1}{x^6}$ when $x-\frac{1}{x}=5$ and $x+\frac{1}{x}=3$.

3. Evaluate the following using formula:

(i) $(103)^2$ **(ii)** $(97)^2$ **(iii)** $(103)\times(97)$

4. Factorize:

(i) $5x+10y+15z$ (ii) $x^2+14x+49$ (iii) $144x^2-121y^2$

(iv) $25x^2-10xy+y^2$ (v) $a^2+4ab+4b^2-9y^2$ (vi) $1-\frac{1}{x^4}$

5. Find the cube of

(i) $2x+3y$ (ii) $3x-4y$ (iii) $x+\frac{1}{x}$ (iv) $x-\frac{1}{x}$

6. Find the cube of:

(i) 29 **(ii)** 31 by using cube formula.

7. Solve the following system of linear equations by

(a) Using substitution method

(i) $2x-y=6$, $x+3y=10$ **(ii)** $2x+5y=1$, $x-2y=4$

(b) Using elimination method

(i) $3x+2y=7$, $2x+3y=8$ **(ii)** $3x-2y=7$, $5x+y=29$

(c) Using cross-multiplication method

(i) $2x+y=5$, $3x-2y=4$ **(ii)** $5x+y=56$, $x+18y=29$.

8. The width of a rectangle is half of its length. Find its area if its perimeter is 24cm.

9. Eliminate x by using formula:

(i) $x-\frac{1}{x}=2k-1$; $x+\frac{1}{x}=5k$ **(ii)** $x-\frac{1}{x}=m$, $x^2+\frac{1}{x^2}=n^2$

10. The sum of the digits of a two-digit number is 5. The digits are interchanged and the new number is greater than the original number by 9. Find the number.

(i) By formation of one linear equation in one variable.

(ii) By formation of two simultaneous linear equation in two variables.

SUMMARY

- The process of expressing a polynomial as a product of two or more polynomials is called factorization.
- Some basic formulas:

  (i) $(a+b)^2 = a^2 + 2ab + b^2$ (ii) $(a-b)^2 = a^2 - 2ab + b^2$

  (iii) $a^2 - b^2 = (a+b)(a-b)$

  (iv) $(a+b)^3 = a^3 + 3ab(a+b) + b^3 = a^2 + 3a^2b + 3ab^2 + b^3$

  (v) $(a-b)^3 = a^3 - 3ab(a-b) - b^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- An equation with degree one is called linear equation.
- A linear equation in one variable can generally be written in the form of $ax + b = 0$, for a, b being real numbers with $a \neq 0$.
- Two or more linear equations which occur in one and the same problem are called simultaneous linear equations.
- A linear equation in two variables has infinite solutions.
- The system of two distinct linear equations in two variables may have one or no solution and can be solved by three methods i.e.
- There are several methods to solve simultaneous linear equations viz. elimination method, substitution method and cross-multiplication method.
- The method of obtaining a relation free of a particular variable is called elimination.

UNIT 7

FUNDAMENTALS OF GEOMETRY

7.1 PARALLEL LINES

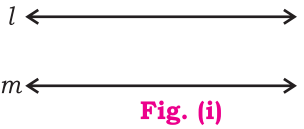


Fig. (i)

7.1.1 Define Parallel Lines

Two lines are said to be parallel, if

(i) They are coplanar and
(ii) They do not intersect.

Such as a 'Railway Track'. Parallel is denoted by '||'.
$\overrightarrow{l} \,||\, \overrightarrow{m}$ is read as '$\overrightarrow{l}$ is parallel to $\overrightarrow{m}$',

AB || CD and EF || GH

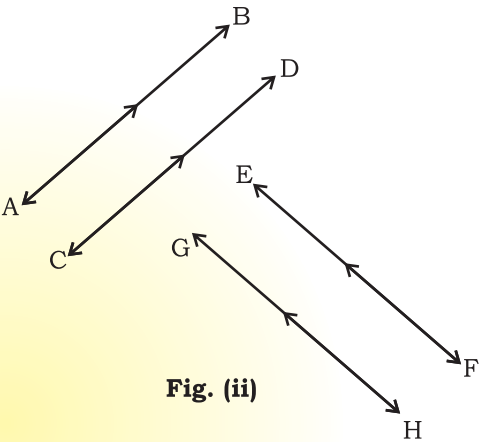


Fig. (ii)

In case of parallel lines, the distance between them is always the same. For example, consider the following figure.

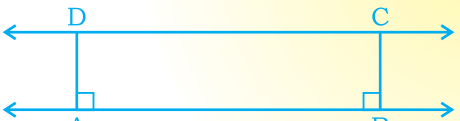


Fig. (iii)

On $\overleftrightarrow{AB}$, $\overline{AD} \perp \overleftrightarrow{AB}$ and $\overline{BC} \perp \overleftrightarrow{AB}$. Hence $m\overline{AD} = m\overline{BC}$. Note that $\overline{AD} \,//\, \overline{BC}$.

7.1.2 Demonstrate through figures the following Properties of Parallel Lines

***Property* 1. Two lines which are parallel to the same given line are parallel to each other.**

Look at the figures given below:

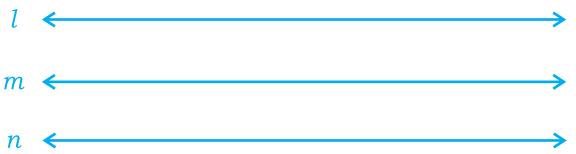


Fig. (iv)

In the figures given above, $\overleftrightarrow{l} \parallel \overleftrightarrow{n}$ and $\overleftrightarrow{m} \parallel \overleftrightarrow{n}$, therefore $l \parallel m$. This is an important property of parallel lines. The following figure also depicts the same property.

Here the top of a kid's table has been placed at the bottom with its legs showing up. Leg AB is parallel to Leg CD, and Leg EF $\parallel$ Leg CD. Identify the pair of legs parallel to each other.

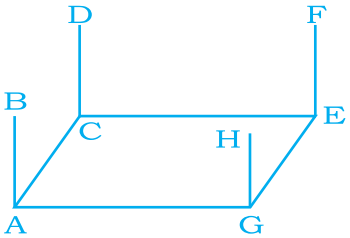


Fig. (v)

***Property* 2. If three parallel lines are intersected by two transversals in such a way that the two intercepts on one transversals are equal to each other, then the two intercepts on the second transversals are also equal.**

Look at the figure given on the right.

There are three lines l, m and n parallel to one another.

Transversal $\overleftrightarrow{AB}$ intersects them at points E, F, G respectively such that $m\overline{EF} = m\overline{FG}$. Another transversal $\overleftrightarrow{CD}$ intersects the three parallel lines at points P, Q, R respectively making intercepts $\overline{PQ}$ and $\overline{QR}$.

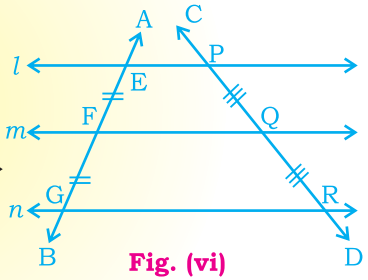


Fig. (vi)

Since intercepts made by the first transversal $\overleftrightarrow{AB}$ are equal in measure, therefore $m\overline{PQ} = m\overline{QR}$, i.e., intercepts on the second transversal are also equal in measure. It should be noted that this property does not say that intercepts on the first transversal are equal to intercepts on the second transversal, but intercepts on the second transversal are equal to one another. These intercepts (on the second transversal) will be either greater or less than the intercepts on the first transversal. These four intercepts will be equal to one another only if the two transversals are parallel to one another, as it can be seen in the figure on the right. Here: $m\overline{EF} = m\overline{FG} = m\overline{PQ} = m\overline{QR}$

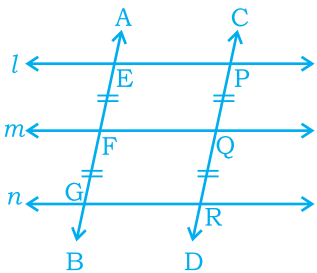


Fig. (vii)

Property 3. **A line through the midpoint of the side of a triangle parallel to another side bisects the third side. (An application of above property)**

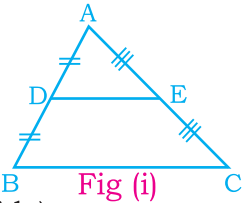


Fig (i)

Look at the figure given on the right.

In ΔABC, D is the midpoint of $\overline{AB}$ and $\overline{DE} \parallel \overline{BC}$

That is: $m\overline{AD} = m\overline{DB}$

Then by Property 2, point E bisects $\overline{AC}$ (the third side)

That is: $m\overline{AE} = m\overline{EC}$

Now look at figure (ii), to see how the above properties have been applied here. Through point A, line $\overleftrightarrow{FG}$ has been drawn parallel to $\overline{DE}$.

Hence by Property 1, $\overleftrightarrow{FG} \parallel \overline{BC}$. and by Property 2, $m\overline{AE} = m\overline{EC}$,

(because $m\overline{AD} = m\overline{DB}$)

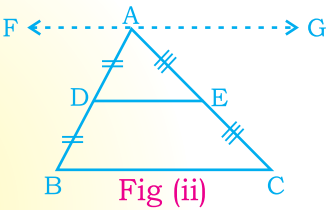


Fig (ii)

7.1.3 Draw a transversal to intersect two parallel lines and demonstrate corresponding angles, alternate interior angles, vertically opposite angles and interior angles on the same side of transversal. Here, we have to demonstrate and observe the Pair of angles formed when a transversal intersects two parallel Lines

When a transversal intersects two or more parallel lines, angles formed are:
(i) Vertically Opposite Angles
(ii) Corresponding Angles
(iii) Alternate Interior Angles
(iv) Interior Angles on the same side of the transversal.

In the figure given here, two parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are intersected by the transversal $\overleftrightarrow{EF}$ as a result, eight angles ($\angle 1, \angle 2, \ldots, \angle 8$) are formed.

Out of these Angles, $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ are **Interior Angles**, whereas $\angle 5$, $\angle 6$, $\angle 7$ and $\angle 8$ are **Exterior Angles**.

($\angle 1$, $\angle 5$); ($\angle 2$, $\angle 6$); ($\angle 3$, $\angle 7$) and ($\angle 4$, $\angle 8$) are four pairs of **Vertically Opposite Angles**.

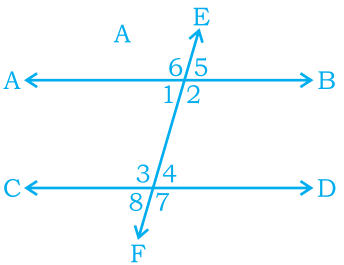


Fig (iii)

Pairs of **Corresponding Angles** are:

(i) $\angle 4, \angle 5$ (ii) $\angle 3, \angle 6$ (iii) $\angle 1, \angle 8$ (iv) $\angle 2, \angle 7$.

Note that in each pair of corresponding angles:

One angle is interior angle, whereas the other is an exterior angle; both angles of a pair are on the same side of the transversal and both angles are formed on different vertices.

Two pairs of **Alternate Interior Angles** are:

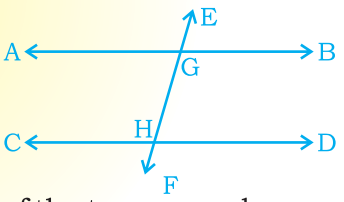
(i) $\angle 1$ and $\angle 4$ (ii) $\angle 2$ and $\angle 3$.

Note that alternate interior angles are formed on the opposite side of transversal as well as on the different vertices.

Two pairs of **Interior Angles on the same side of the transversal** are:

(i) $\angle 1, \angle 3$ and (ii) $\angle 2, \angle 4$

Example 1. If two parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are intersected by a transversal $\overleftrightarrow{EF}$ at points G and H, then write the following pair of angles.



(i) Vertically Opposite angles

(ii) Corresponding angles

(iii) Alternate Interior angles

(iv) Interior Angles on the same side of the transversal

Solution: (i) Four pairs of **Vertically Opposite Angles** are:

(a) $\angle AGH, \angle EGB$ **(b)** $\angle AGE, \angle BGH$ **(c)** $\angle CHF, \angle GHD$ **(d)** $\angle GHC, \angle DHF$

(ii) Four pairs of **Corresponding Angles** are:

(a) $\angle AGH, \angle CHF$ **(b)** $\angle AGE, \angle GHC$ **(c)** $\angle EGB, \angle GHD$ **(d)** $\angle BGH, \angle DHF$

(iii) Two pairs of **Alternate Interior Angles** are:

(a) $\angle AGH, \angle GHD$ **(b)** $\angle BGH, \angle GHC$

(iv) Two pairs of **Interior Angles on the same side of transversal** are:

(a) $\angle AGH, \angle GHC$ (b) $\angle BGH, \angle GHD$

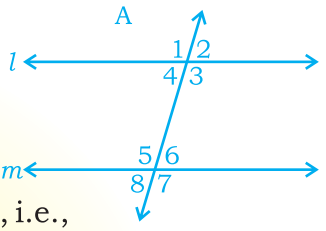
7.1.4. Describe the following Relations between the pairs of angles when a transversal intersects two parallel lines

When a transversal intersects two parallel lines:

(i) Pairs of corresponding angles are equal

(ii) Pairs of alternate interior angles are equal

(iii) Pairs of interior angles on the same side of a transversal are supplementary.



Consider the figure in which l is parallel to m and n intersects them.

(a) The pairs of **Corresponding Angles** are:

$(\angle 1, \angle 5); (\angle 2, \angle 6); (\angle 4, \angle 8); (\angle 3, \angle 7)$

All these pairs of angles are equal in measure, i.e.,

$m\angle 1 = m\angle 5; m\angle 2 = m\angle 6; m\angle 4 = m\angle 8; m\angle 3 = m\angle 7$

(b) The pairs of **Alternate Interior Angles** are:

$\angle 3$ and $\angle 5$; $\angle 4$ and $\angle 6$

All these pairs of angles are equal in measure, i.e.,

$m\angle 3 = m\angle 5; m\angle 4 = m\angle 6$

(C) The pairs of **Interior Angles on the same side of the transversal** are:

$\angle 5$ and $\angle 4$; $\angle 6$ and $\angle 3$.

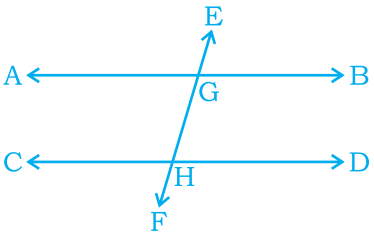
All these pairs of angles are supplementary, i.e.,

$\angle 5$ and $\angle 4$ are supplementary; $\angle 6$ and $\angle 3$ are supplementary.

It implies that : $m\angle 5 + m\angle 4 = 180^\circ$

and $m\angle 6 + m\angle 3 = 180^\circ$

Example 2. In the figure on the right, identify the angle which is:



(i) Congruent to $\angle GHD$ being corresponding angle to it.

(ii) Congruent to $\angle GHD$ being an alternate angle of it.

(iii) Supplement of $\angle GHD$ being

(a) an interior angle on the same side of the transversal

(b) an interior angle adjacent to it

(c) an exterior angle adjacent to it

(iv) Congruent to $\angle GHD$ being its vertically opposite angle

Solution: (i) $\angle EGB \cong \angle GHD$ (Corresponding angles)

(ii) $\angle AGH \cong \angle GHD$ (Alternate angles)

(iii) (a) $\angle BGH$ is supplement of angle GHD
(Interior angle on the same side of the transversal)

(b) $\angle GHC$ is supplement of angle GHD
(Interior angle adjacent to it)

(c) $\angle DHF$ is supplement of $\angle GHD$
(Exterior angle adjacent to it)

(iv) $\angle CHF \cong \angle GHD$ (Vertically opposite angles)

Example 3. In the figure of Example 2, find the remaining angles when $m\angle GHD = 70^\circ$

Solution:

(1) $\angle AGH \cong \angle GHD$ (Alternate angles) $\therefore\ m\angle AGH = 70^\circ$

(2) $\angle EGB \cong \angle GHD$ (Corresponding angles) $\therefore\ m\angle EGB = 70^\circ$

(3) $m\angle BGH + m\angle GHD = 180^\circ$
(Interior angle on the same side of transversal)

$\Rightarrow\quad m\angle BGH = 180^\circ - 70^\circ = 110^\circ$

(4) $\angle AGE \cong \angle BGH$ (Vertical Opposite angles) $\therefore\ m\angle AGE = 110^\circ$

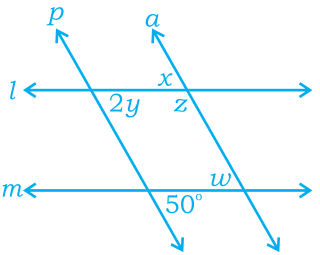
(5) $\angle GHC \cong \angle BGH$ (Alternate angles) $\therefore\ m\angle GHC = 110^\circ$

(6) $\angle DHF \cong \angle GHC$ (Vertical Opposite angles) $\therefore\ m\angle DHF = 110^\circ$

(7) $\angle CHF \cong \angle GHD$ (Vertical Opposite angles) $\therefore\ m\angle CHF = 70^\circ$

Example 4. If $l \parallel m$ and $p \parallel q$, then find the values of the measures of angles w, x, y and z

Solution:



(1) w is an alternate angle of 50°

$\therefore\ m\angle w = 50^\circ$

(2) x and w are corresponding angles

$\therefore\ m\angle x = m\angle w \Rightarrow m\angle x = 50^\circ$

(3) $\angle 2y$ is an alternate angle of x

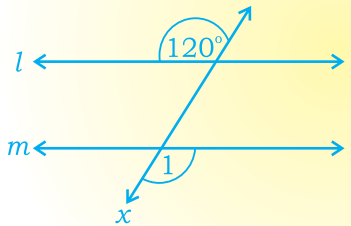
$\therefore\ m\angle 2y = m\angle x \Rightarrow m\angle 2y = 50^\circ$ or $m\angle y = \frac{50^\circ}{2} = 25^\circ$

(4) $\angle z$ is supplement of $\angle x \Rightarrow m\angle z = 180^\circ - 50^\circ = 130^\circ$

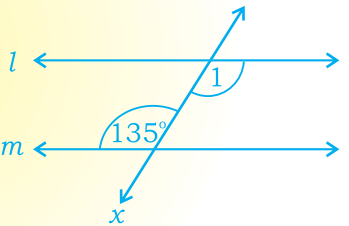
EXERCISE 7.1

1. Find the measure of $\angle 1$ in each of the figures below when measure of one angle is given.

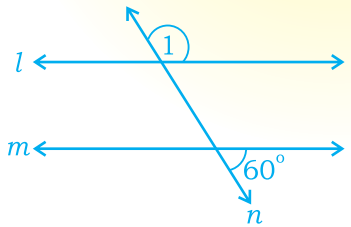
(a)



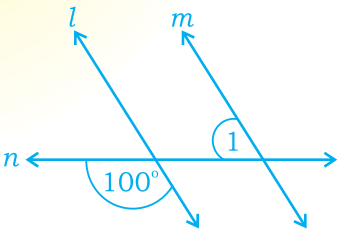
(b)



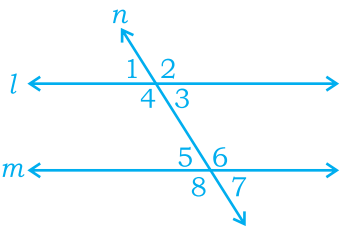
(c)



(d)



2. In the adjacent figure $l // m$.
Find the measure of remaining angles when $m\angle 3 = 65^\circ$.



3. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ intersects them at points P and Q respectively, $m\angle PQD = 50^\circ$. Draw the figure and find the measure of remaining angles.

4. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ intersects them at points G and H respectively. Identify the angle which is:

(i) Congruent to $\angle GHC$ as a corresponding angle

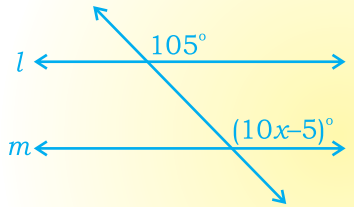
(ii) Congruent to $\angle GHC$ as an alternate angle

(iii) Congruent to $\angle GHC$ as a vertically opposite angle

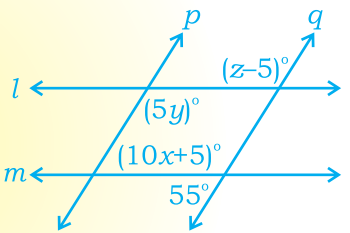
(iv) Supplement of $\angle GHC$: (a) as an interior angle on the same side of the transversal (b) as an interior angle adjacent to it (c) as an exterior angle adjacent to it.

5. Solve for x, y or z and find the angle in each case when $l \parallel m$ and $p \parallel q$.

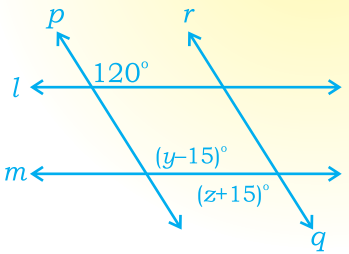
(i)

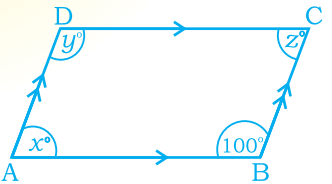


(ii)



(iii)



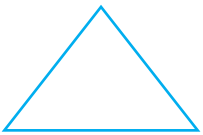


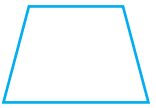
6. $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$,
Find the measure of angles x, y, z.

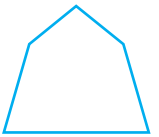
7.2 POLYGONS

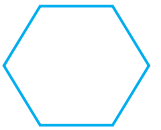
7.2.1 Define of a Polygon

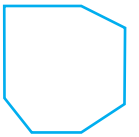
A plane closed figure bounded by three or more sides is called a **Polygon**. Figures of some important polygons along with their names are given here.


Triangle
(Tri means three)


Quadrilateral
(Quad means four)

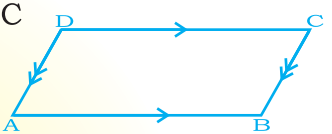

Pentagon
(Penta means five)


Hexagon
(Hexa means six)


Heptagon
(Hepta means seven)

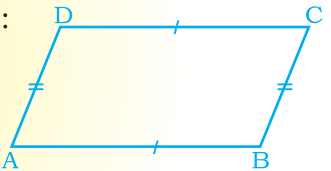
7.2.2 Demonstrate the following Properties of a parallelogram

Opposite sides of a parallelogram are equal. Diagonals of a parallelogram bisect each other. We already know that "Parallelogram is a quadrilateral whose opposite sides are parallel." For example, the figure shown here is a quadrilateral because it has four sides, but since $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$.



$\therefore$ Quad. ABCD is a parallelogram. Symbolically a parallelogram is denoted as $\parallel^m$, i.e. the above figure on the right is $//^m$ ABCD.

A parallelogram has the following properties:



1. Opposite sides are equal.

Opposite sides of a parallelogram are congruent, e.g., *In* $\parallel^m$ ABCD.

$\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{BC}$

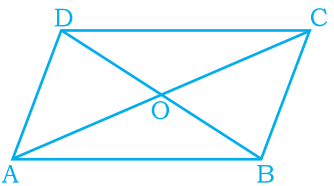
2. Opposite angles are congruent.

Opposite angles of a $\parallel^m$ are congruent, e.g., *In* $\parallel^m$ ABCD

$\angle A \cong \angle C$ and $\angle B \cong \angle D$

3. Diagonals bisect each other.

Diagonals of a $\parallel^m$ bisect one other. For example,



In $\parallel^m$ ABCD, diagonal $\overline{AC}$ intersects diagonal $\overline{BD}$ at O in such a way that O is the mid-point. It implies that $\overline{AC}$ and $\overline{BD}$ bisect one an other, i.e., $\overline{AO} \cong \overline{CO}$ and $\overline{BO} \cong \overline{DO}$ and

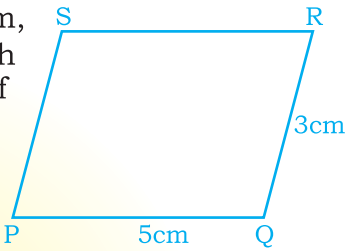
$m\,\overline{AO} = \frac{1}{2}\, m\overline{AC} = m\,\overline{CO}$ and $m\,\overline{BO} = \frac{1}{2}\, m\,\overline{BD} = m\,\overline{DO}$.

4. Angles at the end-points of each side are supplementary

Each pair of consecutive angles (i.e., angles at the end- point of each side) of a $||^m$ are supplementary. e.g., *In* $||^m$ ABCD, $m\angle A + m\angle B = 180^\circ$. Similarly.

$m\angle B + m\angle C = m\angle C + m\angle D = m\angle D + m\angle A = 180^\circ$.

Example 1. *In* $||^m$ PQRS, $m\overline{PQ}$ = 5cm, $m\overline{QR}$ = 3cm and $m\angle P = 70^\circ$, find the length of other two sides and measure of remaining angles.



Solution:

$\overline{RS} \cong \overline{PQ}$ (opposite sides of a $||^m$) and m $\overline{PQ}$ = 5cm

$\therefore$ $m\overline{RS}$ = 5cm

$\overline{PS} \cong \overline{QR}$ (opposite side of a $||^m$) and $m\overline{QR}$ = 3cm

$\therefore$ $m\overline{PS}$ = 3cm

$\angle R \cong \angle P$ (opposite angles of a $||^m$) and $m\angle P = 70^\circ$

$\therefore$ $m\angle R = 70^\circ$

Further $m\angle P + m\angle Q = 180^\circ$ (Int. $\angle$s on the same side of transversal)

$\Rightarrow m\angle Q = 180^\circ - 70^\circ$ ($\because$ $m\angle P = 70^\circ$)

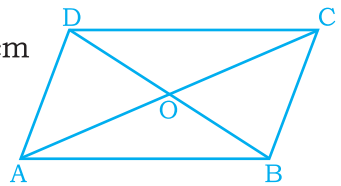
$\Rightarrow m\angle Q = 110^\circ$

Also $m\angle S = m\angle Q$ (opposite angles of a $||^m$) and $m\angle Q = 110^\circ$

$\therefore$ $m\angle S = 110^\circ$

Example 2. *In* $||^m$ ABCD, $m\overline{BD}$ = 4cm and $m\overline{AC}$ is $\frac{3}{2}$ times of $m\overline{BD}$.

Both diagonals intersect at O. Find the measure of $\overline{OC}$ and $\overline{OD}$.



Solution: $m\overline{OD} = \frac{1}{2} m\overline{BD} = \frac{1}{2}$ (4cm) = 2cm

$\Rightarrow$ $m\overline{OD}$ = 2cm

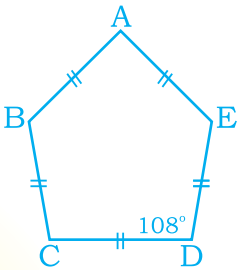
Further $m\overline{AC} = \frac{3}{2}$ of $m\overline{BD} = \frac{3}{2} \times 4 = 6$cm

$\therefore$ $m\overline{OC} = \frac{1}{2} m\overline{AC} \Rightarrow m\overline{OC} = \frac{1}{2}$ (6) = 3cm

7.2.3. Define Regular Pentagon, Hexagon and Octagon

A polygon in which all the sides are congruent (equal in measure) to one another is called a **Regular Polygon**. In a regular polygon all interior angles are congruent to one another.

Regular Pentagon.



A pentagon (i.e., five sided closed figure) in which all sides are congruent to each other is called a Regular Pentagon.

In a regular pentagon each angle is also congruent to one another.

Sum of measures all the angles of a regular polygon is

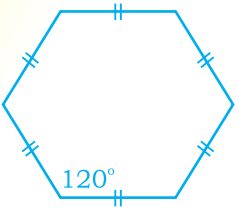
$$S = (2n-4)\ rt\angle s$$

$$= 2(5)-4 = (10-4)\ rt\angle s = 6rt\angle s = 6\times 90 = 540^\circ$$

$\therefore$ Measure of each angle of a Regular Pentagon $= \frac{540}{5} = 108^\circ$

Regular Hexagon.

A hexagon (six-sided closed figure) in which all sides are congruent to each other is called a **Regular Hexagon.**



In a regular hexagon each angle is also congruent to one another.

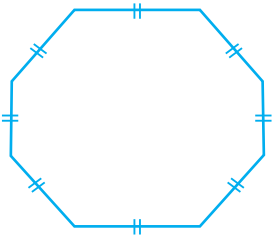
Sum of measure of all the angles of a Hexagon is:

$$S = (2n-4)\ rt\angle s = (2\ (6)-4)\ rt\angle s$$

$$= (12-4)\ rt\angle s = 8\ rt\angle s = 8\times 90 = 720^\circ$$

Each angle of a Regular Hexagon measures $= \frac{720^\circ}{6} = 120^\circ$.

Regular Octagon.



An octagon (eight sided closed figure) all of whose sides are congruent to one another is called a Regular Octagon.

In a regular octagon each angle is also congruent to one another.

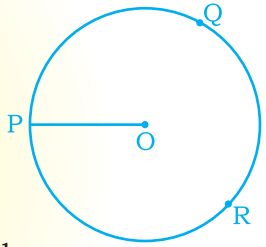
Sum of measures of all the angles of a Regular Octagon is: $S = (2n-4)\ rt\angle s$

$$S = (2(8)-4)\ rt\angle s = 16-4 = 12\ rt\angle s = 12 \times 90^\circ = 1080^\circ$$

$\therefore$ Measure of each angle of a regular octagon $= \frac{1080^\circ}{8} = 135^\circ$.

7.3 CIRCLE

A circle is a simple closed curve all of whose points are equidistant from a fixed point called its centre. Alternatively, a circle is a set of points in a plane equidistant from a fixed point called its centre.



Here PQR is a circle whose centre is O. A circle is denoted by naming any of its three points or simply we write $\odot$ O (Circle O). $\overline{OP}$ is the radial segment and $m\overline{OP}$ is the radius of the circle.

7.3.1 Demonstrate a point lying in the interior and exterior of a circle.

When a circle is drawn on a plane, the set of points of the plane has three subsets, viz;

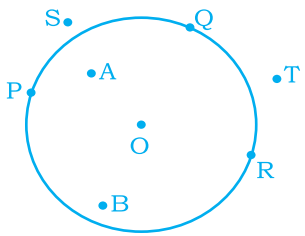
(1) **Set of points of the circle:** It consists of all these points whose distances from the centre are equal to its radius. Points P, Q, R belong to this set. This set of points is called the boundary of the circle.

(2) **Set of points lying inside the circle:** It consists of all the parts whose distances from the centre are less than its radius. This set of points is called the Interior of the Circle .

Points A and B are in the interior of the circle because

$$m\overline{OA} < m\overline{OP} \text{ or } m\overline{OQ} \text{ or } m\overline{OR}$$
(radius of the circle)

Similarly $m\overline{OB} < m\overline{OP}$



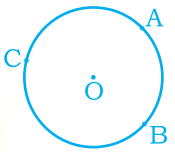
(3) Set of points lying outside the circle. It consists of all the points whose distances from the centre are greater than its radius. This set of points is called the Exterior of the Circle. In the above figure, points S and T are in the exterior of the circle because

$$m\overline{OS} > m\overline{OP} \text{ or } m\overline{OQ} \text{ or } m\overline{OR} \quad \text{(radius of the circle)}$$

Similarly $m\overline{OT} > m\overline{OP}$

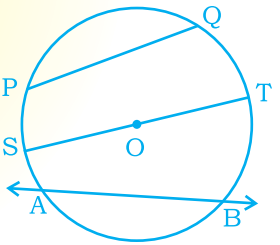
Note that the set of points of the Interior of the circle and the points on the boundary of circle (i.e. union of the two sets) is called Circular Region. Circle and Circular Region are two different geometrical entities and mathematicians do not use them interchangeably.

7.3.2. Describe the terms; sector, secant and chord of a circle, noncyclic points, tangent to a circle and concentric circles.



The description of terms related to a circle are as under:

Arc: Any portion or part of the boundary of a circle is called an arc of the circle. Arc may be minor or major. An arc is said to be a Minor Arc if it is less than a Semi-circle (Half Circle). An arc is said to be a Major Arc if it is greater than a semi-circle. In the above figure $\overset{\frown}{AB}$ is a minor arc whereas $\overset{\frown}{ACB}$ is a major arc.



Chord: A line segment joining any two points of a circle is called a chord of a circle. A chord passing through the centre is called a Diameter.

Measure of a Diameter = 2 radius

Here $\overline{PQ}$ is a chord. $\overline{ST}$ is also a chord but since it is passing through the centre O, therefore it is called a diameter.

Secant: A line which intersects the circle at any two of its points is called a secant. In the above figure $\overleftrightarrow{AB}$ is a secant.

Note that $\overline{AB}$ (a subset of $\overleftrightarrow{AB}$) is a chord whereas $\overleftrightarrow{AB}$ is a secant.

Tangent: A line (line segment or ray) is called a tangent to the circle if it intersects the circle at one and only one point, i.e., it touches the circle at one point only.

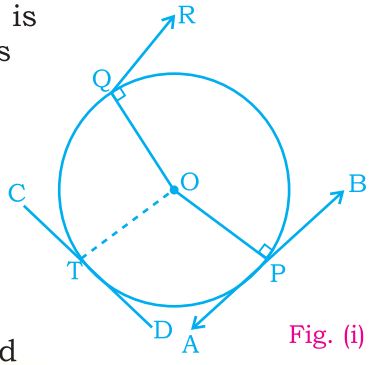


Fig. (i)

In figure (i) $\overleftrightarrow{AB}$, $\overrightarrow{QR}$ and $\overline{CD}$ are tangents to the $\odot$O at points P, Q and T. Points P, Q and T are called Points of Tangency. Radial Segment is always perpendicular to the tangent at the point of tangency, i.e., $\overline{OP} \perp \overleftrightarrow{AB}$, $\overline{OQ} \perp \overrightarrow{QR}$ and $\overline{OT} \perp \overline{CD}$.

Sector: A portion of circular region bounded by two radial segments and an arc is called a sector.

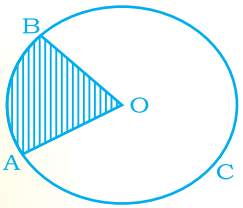


Fig. (ii)

In figure (ii), the shaded portion is a sector of a circle whose centre is O. This sector is bounded by radial segments $\overline{OA}$ and $\overline{OB}$ and minor arc $\overset{\frown}{AB}$. Note that the unshaded portion is also a sector. It is bounded by $\overline{OA}$, $\overline{OB}$ and major are $\overset{\frown}{ACB}$.

Concyclic Points: If four or more points are such that a circle passes through all the four (or more) points, these points are called concyclic points.

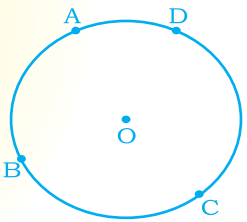


Fig. (iii)

In figure (iii) points A, B, C and D are concyclic points. Vertices of a rectangle and square are always concyclic because their opposite angles are supplementary.

In figure (iv) points A, B, C, D are concyclic. So that $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$.

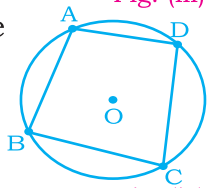


Fig. (iv)

Concentric circles: Two or more circles having the same centre and different radii are called concentric circles.

In figure (v) $\odot$ ABC, $\odot$ PQR and $\odot$ XYZ are concentric circles.

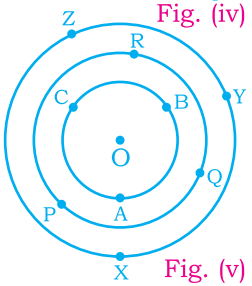


Fig. (v)

EXERCISE 7.2

1. Select 'False' statements out of the following statements.

(i) A rectangle is a parallelogram.

(ii) A square is a parallelogram as well as a rectangle.

(iii) In a $\|^m$ interior angles on the same side of a transvarsal are supplementary.

(iv) If a transversal intersects two lines in the same plane, then the alternate interior angles are not always congruent.

(v) If the interior angles on the same side of a transversal are supplementary, the transversal intersects two lines in the same plane which do not intersect.

(vi) If the diagonals of a quadrilateral bisect each other at right angles, the quadrilateral is a parallelogram.

(vii) In a rectangle, the diagonals do not bisect each other at right angles.

(viii) In a $\|^m$, if a diagonal is drawn, it makes two pairs of alternate angles, each pair consisting of congruent angles.

(ix) A square is a regular quadrilateral.

(x) If two lines never intersect however long they are produced, still they may or may not be parallel.

2. Perimeter of a $\|^m$ PQRS is 18cm. If one side is twice that of the other, find the measure of each side.

3. In $\|^m$ ABCD, sum of measures of one pair of opposite angles is 130°, find the measure of each angle.

4. In $\|^m$ WXYZ, if $m\angle X$ is twice of $m\angle W$, find the measure of each angle.

5. In a $||^m$ ABCD, diagonal $\overline{AC}$ is twice that of $\overline{BD}$. It the sum of the measures of both the diagonals is 9cm, find the measure of each diagonal. If diagonals intersect at point P, find $m\,\overline{PA}$ and $m\,\overline{PB}$ and express the relation $m\,\overline{PA} : m\,\overline{PB}$.

6. Define the following:

(i) Parallelogram **(ii)** Regular Pentagon **(iii)** Regular Octagon

(iv) Chord **(v)** Secant **(vi)** Sector

(vii) Tangent **(viii)** Concyclic points **(ix)** Concentric circles

(x) Diameter

7. The radius of a circle whose centre is O is 3cm long. Three points P, Q, R are in the same plane such that the distance of P from O is 2.5cm, the distance of point Q from O is 4cm and the distance of R from O is 3cm. Discuss the location of each point with reference to the given circle.

8. Select 'True' statements out of the following given statements. Rewrite the statement correctly if it is 'False'.

(i) A diameter passing through the centre of a circle is called a chord.

(ii) A portion of circular region enclosed by two radii and one sector is called an arc.

(iii) Secant is a line segment intersecting a circle at two points.

(iv) A point lies in the exterior of a circle if its distance from the circle is greater than its radius.

(v) Three or four points are said to be concyclic if a circle can pass through those points.

(vi) Vertices of a $||^m$ are concylic.

(vii) If four points are concyclic, opposite angles of the quadrilateral formed by these points are complementary.

(viii) Arc of a circle is not a subset of the circle.

(ix) Radius of a circle is twice of its diameter.

(x) A line parallel to the tangent at the point of tangency passes through the centre of the circle.

REVIEW EXERCISE 7

1. Fill in the blanks with appropriate words.

(i) Two_____lines are said to be parallel if they do not intersect.

(ii) Two lines which are parallel to the same line are _____ to one another.

(iii) A line through the _____ of one side of a triangle and parallel to another side bisects the _____ side.

(iv) Pairs of angles formed when a transversal intersects two parallel lines are (a) Corresponding angles (b) _____ interior angles and (c) _____ angles on the _____ of _____.

(v) When a transversal intersects two parallel lines (a) Pairs of _____ and _____ angles are congruent (b) pairs of interior _____ on the same _____ of transvarsal are _____.

(vi) Polygon is a closed figure in a_____ bounded by _____ or _____ sides.

(vii) Parallelogram is a quadrilateral whose opposite sides are _____ and _____.

(viii) A quadrilateral in which diagonals bisects one another is a _____.

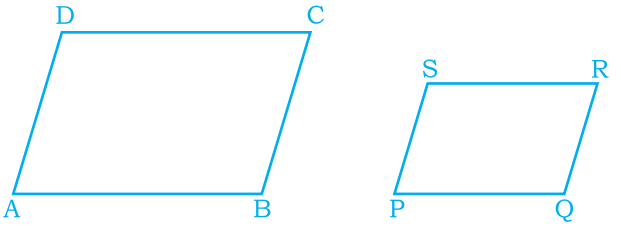
(ix) In a $||^m$, _____ angles at the end point of each side are supplementary.

(x) A polygon all of whose sides are congruent is called a_____.

(xi) A _____ is a set of points equidistant from a fixed point called its centre.

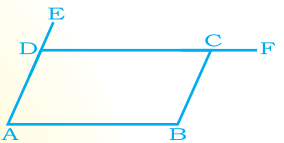
(xii) A line segment joining any two points of a circle is called a _____.

2. In the two parallelograms given below:



$\overline{PS} \parallel \overline{AD}$ and $\overline{RS} \parallel \overline{CD}$, $m\,\overline{PS}$ =3cm, $m\,\overline{RS}$ = 4cm and $m\angle P = 70^\circ$. Find measures of the remaining angles of $\parallel^m$ PQRS. If sides of $\parallel^m$ ABCD are one and half times of the sides of $\parallel^m$ PQRS, find the sides and angles of $\parallel^m$ PQRS. Is $m\angle A = \frac{3}{2}\, m\angle P$?

3. In $\parallel^m$ ABCD, $\overline{AD}$ has been produced to E and $\overline{DC}$ to F and $m\angle CDE = 65^\circ$. Find the measures of remaining angles and write all pairs of:



(i) Corresponding angles **(ii)** Alternate angles and

(iii) All pairs of supplementary angles

4. Define the following terms:

(i) Parallel lines **(ii)** Parallelogram **(iii)** Arc **(iv)** Secant

(v) Tangent **(vi)** Concyclic Points **(vii)** Concentric Circle

5. Define the following polygons and determine the measure of interior angles in each case.

(a) Regular Pentagon **(b)** Regular Hexagon **(c)** Regular Octagon

6. Answer the following questions:

(i) The radius of a circle is 4cm. How long is the diameter of this circle?

(ii) Three concentric circles have been drawn whose centre is O. The diameters of these circles are respectively 5cm, 7cm and 10cm. Find the radii of each circle.

(iii) How many circles can be drawn **(a)** through three non-collinear points **(b)** through four concyclic points?

(iv) How many diameters can be drawn for a given circle?

SUMMARY

- Two coplanar lines which do not intersect are called parallel lines.
- Two lines which are parallel to the same line are parallel to one another.
- If three parallel lines are intersected by two transvarsals in such a way that the two intercepts on one transvarsal are equal to each other, then the two intercepts on the second transvarsal are also equal.
- A line through the mid-point of one side of a triangle parallel to another side bisects the third side.
- When a transversal intersects two prallel lines then:
  - **(i)** Corresponding angles are congruent
  - **(ii)** Alternate Interior angles are congruent
  - **(iii)** Interior angles on the same side of a transvarsal are supplementary.
- A parallelogram is a quadrilateral whose opposite sides are parallel.
- A polygon is a closed plane figure bounded by three or more line segments.
- In a Regular Polygon all sides and all angles are congruent.
- A circle is a set of points in a plane all of whose points are equidistant from a fixed point.
- Chord is a line segment whose end points lie on a circle.
- A Secant is a line intersecting the circle at two points.
- Sector is a part of a circular region bounded by two radial segments and an arc.
- Four or more points that lie on a circle are called concyclic points.
- Two or more circles having a common centre but different radii are called concentric circles.
- A line (ray or line segment) is called a tangent if it intersects/ touches the circle only at one point.

UNIT 8

PRACTICAL GEOMETRY

8.1 CONSTRUCTION OF QUADRILATERALS

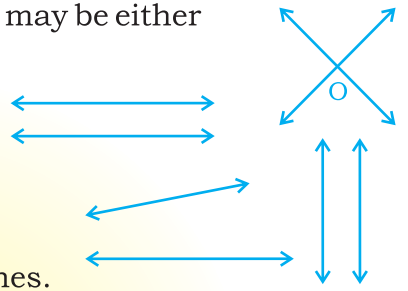
8.1.1 Define and depict two converging (non-parallel) lines and find the angle between them without producing the lines

When we draw two lines in a plane, they may be either

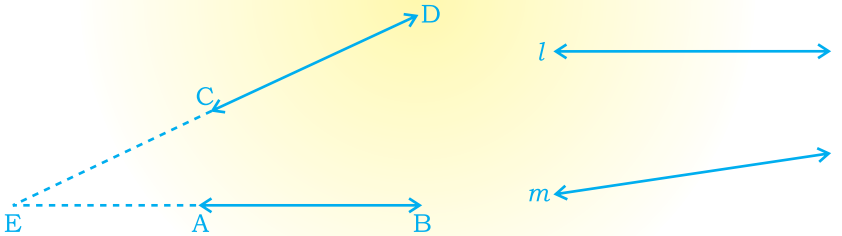
(i) intersecting lines at a point

or (ii) parallel lines

or (iii) neither intersecting nor parallel but in such a state that they will intersect when produced. Such two lines are called converging lines.



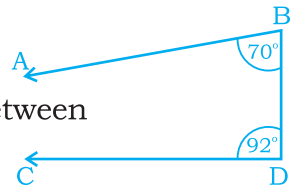
Definition: Two lines or line segments in a plane which are neither intersecting nor parallel but they will intersect at a point when produced are called converging lines.



In the above figure $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are converging lines because when $\overline{DC}$ and $\overline{BA}$ are produced they will meet at a point (say E) making an angle ($\angle AEC$). This phenomenon has been shown by dotted lines.

Similarly l and m will intersect when produced on the converging side. It should be noted that in converging lines one side is converging (getting closer and closer) whereas the other side is non-converging or diverging (i.e. getting farther and farther away). In the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, the side with points A and C is converging whereas the side with points B and D is diverging (or non-converging).

To find the measure of an angle between two converging lines without producing them:

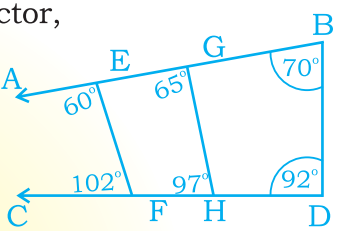


Given: Two converging lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$

Required: To find the measure of the angle between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ without producing them.

Steps of construction:

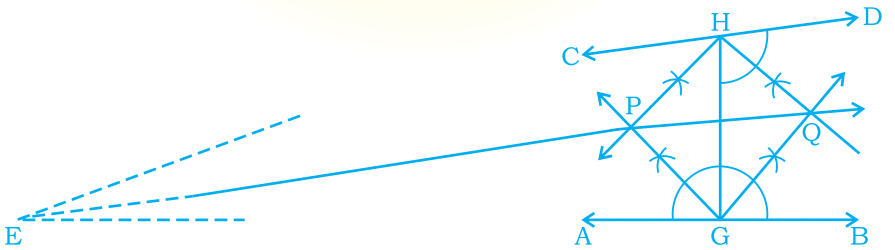
1. Join B to D (Points on the diverging side)
2. Measure $\angle ABD$ with the help of a protractor, we have $m\angle ABD = 70^\circ$
3. Measuring $\angle CDB$ with a protractor, we have $m\angle CDB = 92^\circ$
4. Therefore, the measure of angle (say x) between the converging lines is:



$$m\angle x = 180^\circ - (m\angle ABD + m\angle CDB)$$
$$= 180^\circ - (70^\circ + 92^\circ)$$
$$= 180^\circ - 162^\circ = 18^\circ$$

Note that in whatever way, we join the points on the diverging side of two lines, sum of measures of both the angles will be the same as shown by $\overline{EF}$ and $\overline{GH}$.

8.1.2 Bisect the angle between two converging lines without producing them



Given: Two converging lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$

Required: To bisect the angle (that would form on producing the converging sides of lines) without producing them.

Steps of construction:

1. At a convenient place, draw a transversal intersecting $\overleftrightarrow{AB}$ at G and $\overleftrightarrow{CD}$ at H.
2. Draw the bisectors of $\angle AGH$ and $\angle CHG$ intersecting each other at point P.
3. Draw the bisectors of $\angle BGH$ and $\angle DHG$ intersecting each other at point Q.
4. Draw $\overline{QP}$ and produce it to any point E.

$\overleftrightarrow{QP}$ is the required bisector of the angle between two converging lines without producing them.

8.1.3 Construct a square when:

(a) Measure of its diagonal is given.

(b) Difference between the measures of a diagonal and a side is given.

(c) Sum of the measures of a diagonal and a side are given.

(a) Construct a square when measures of its diagonal is given

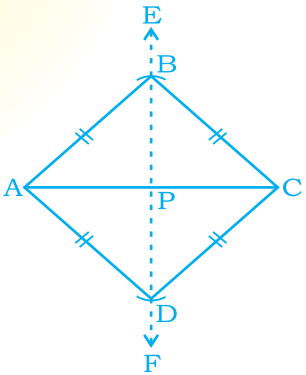
Example 1. Draw square ABCD whose diagonal measures 6cm.

Given: In square ABCD, $m\overline{AC}$ = 6 cm

Required: To construct a square ABCD from the given data

Steps of construction:

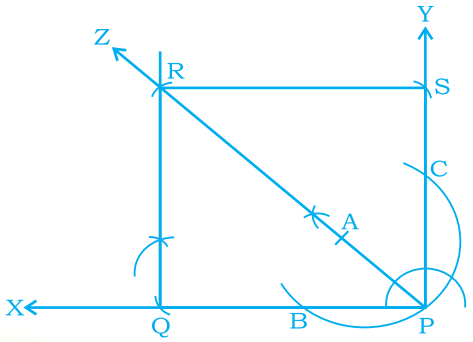
1. Draw $\overline{AC}$, such that $m\overline{AC}$ = 6cm
2. Draw $\overleftrightarrow{EF}$, the right bisector of $\overline{AC}$ intersecting it at point P.
3. Take P as centre and with radius $m\overline{PA}$ (or $m\overline{PC}$) draw arcs on both sides of $\overline{AC}$ cutting $\overleftrightarrow{EF}$ at B and D.
4. Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. Now ABCD is the required square.



(b) Construct a square when difference between the measures of a diagonal and a side is given.

Example 2. Construct square PQRS when the difference between its diagonal and a side is 3 cm.

Given: In Square PQRS, difference between its diagonal and one side is 3 cm



Required: To construct square PQRS from the given data

Steps of construction:

1. Draw a ray $\overrightarrow{PX}$.
2. At point P draw $\overrightarrow{PY} \perp \overrightarrow{PX}$.
3. Draw $\overrightarrow{PZ}$, the bisector of $\angle XPY$
4. From $\overrightarrow{PZ}$, cut off $\overrightarrow{PA}$ such that $m\overline{PA} = 3$ cm
5. Taking point A as centre and with radius 3cm, draw an arc cutting $\overrightarrow{PX}$ at B and PY at point C.
6. Taking points B and C as cnetres and with the same radius (i.e., 3cm) draw arcs cutting $\overrightarrow{PX}$ at Q and $\overrightarrow{PY}$ at point S.
7. Taking points Q and S as centres and with radius m $\overline{PQ}$ (or $m\overline{PS}$) draw arcs cutting one another and $\overrightarrow{PZ}$ at point R.
8. Draw $\overline{QR}$ and $\overline{SR}$. Now PQRS is the required square.

Note: In diagonal $\overline{PR}$, $m\overline{AR} = m\overline{PQ}$ (or $\overline{PS}$ etc.) side of square and $m\overline{AP} = 3$cm (difference between diagonal and side)

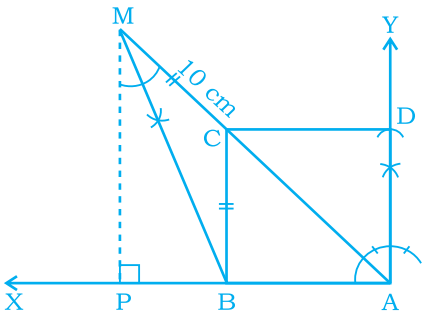
[After the construction is complete, verify the lengths of the diagonal and the side].

(c) Construct a square when the sum of measures of its diagonal and side is given.

Example. Construct square ABCD, when the sum of its diagonal and one side is 10 cm.

Given: In square ABCD, the sum of its diagonal and a side is 10 cm.

Required: To construct square ABCD from the given data.



Steps of construction:

1. Draw a ray $\overrightarrow{AX}$
2. Draw $\overrightarrow{AY} \perp \overrightarrow{AX}$
3. Draw $\overline{AM}$ the bisector of $\angle XAY$ such that $m\overline{AM} = 10$ cm
4. Draw $\overline{MP} \perp \overrightarrow{AX}$
5. Draw $\overline{MB}$, the bisector of $\angle AMP$ meeting $\overrightarrow{AP}$ at point B.

6. Through point B, draw $\overline{BC}$ // $\overline{MP}$ meeting $\overline{AM}$ at point C.
7. From $\overline{AY}$, cut off $\overline{AD} \cong \overline{BC}$
8. Draw $\overline{CD}$.

Now quadrilateral ABCD is the required square.

Note: Sum of a diagonal and a side of square = 10 cm, it has been divided into two parts
(i) diagonal $\overline{AC}$ and (ii) $m\overline{MC} = m\overline{BC}$ (side of square)

8.1.4 Construct a Rectangle

(a) When two sides are given

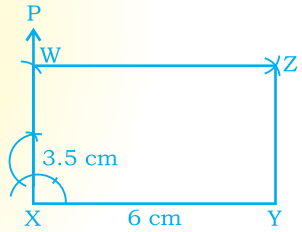
Example 1.

Construct rectangle XYZW in which $m\overline{XY}$ = 6 cm and $m\overline{YZ}$ = 3.5 cm

Given: $m\overline{XY}$ = 6 cm, $m\overline{YZ}$ = 3.5 cm, are sides of a rectangle XYZW.

Required: To construct rectangle XYZW from the given data.

Steps of construction:



1. Draw $\overline{XY}$, 6 cm long
2. At point X, draw $\overrightarrow{XP} \perp \overline{XY}$
3. Take point X as centre and with radius 3.5 cm, draw an arc to cut $\overrightarrow{XP}$ at W.
4. Take point W as centre and with radius $m\overline{XY}$ draw an arc.
5. Take point Y as centre and with radius $m\overline{XW}$ draw an arc which cuts the previous arc at Z.
6. Draw $\overline{WZ}$ and $\overline{YZ}$.

Now XYZW is the required rectangle.

(b) When the diagonal and a side are given.

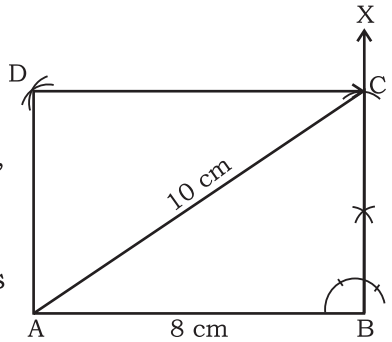
Example 2.

Construct rectangle ABCD in which diagonal m $\overline{AC}$ = 10 cm and a side $\overline{AB}$ = 8 cm.

Given: In rectangle ABCD $m\overline{AC}$ = 10 cm and $m\overline{AB}$ = 8 cm.

Required: To construct a rectangle ABCD from the given data.

Steps of construction



1. Draw $\overline{AB}$, 8cm long
2. At point B, draw $\overrightarrow{BX} \perp \overline{AB}$
3. Take point A as centre, with radius 10 cm, draw an arc which cuts $\overrightarrow{BX}$ at C.
4. Join point A to C (Draw $\overline{AC}$).
5. Take point A as centre, with radius m $\overline{BC}$, draw an arc.
6. Take point C as centre, with radius m $\overline{AB}$, draw another arc which cuts the previous arc at point D.
7. Draw $\overline{AD}$ and $\overline{CD}$.

Now ABCD is the required rectangle.

8.1.5 Construct a Rhombus

(a) When one side and the base angle are given.

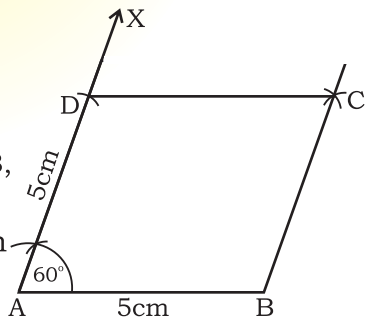
Example 1.

Construct Rhombus ABCD whose sides are 5 cm each and measure of base angle is 60°.

Given: In Rhombus ABCD, $m\overline{AB}$ = 5 cm, $m\angle A = 60^\circ$.

Required: To construct Rhombus ABCD from the given data.

Steps of construction



1. Draw $\overline{AB}$, 5 cm long
2. At point A, construct $\angle BAX$ such that $m\angle BAX = 60^\circ$
3. Taking point A as centre, with radius $m\overline{AB}$, draw an arc to intersect $\overrightarrow{AX}$ at point D
4. Taking points B and D as centres, with the same radius of $m\overline{AB}$, draw arcs cutting each other at C.
5. Draw $\overline{BC}$ and $\overline{DC}$.

Now Quad. ABCD is the required Rhombus.

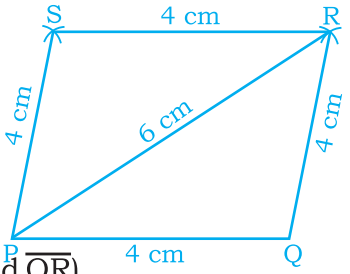
(b) When one side and a diagonal are given.

Example 2. Construct Rhombus PQRS when $m\overline{PQ}$ = 4 cm and $m\overline{PR}$ = 6 cm.

Given: In Rhombus PQRS, $m\overline{PQ} = 4$ cm, $m\overline{PR} = 6$ cm.

Required: To construct Rhombus PQRS from the given data.

Steps of construction:



1. Draw $\overline{PQ}$, 4 cm long.
2. Take point P as centre, with radius 6 cm draw an arc.
3. Take point Q as centre, with radius $m\overline{PQ}$, draw another arc to cut the previous arc at point R.
4. Join point P to R and Q to R. (Draw $\overline{PR}$ and $\overline{QR}$)
5. Taking points P and R as centres and with radius $m\overline{PQ}$. draw arcs to cut each other at point S.
6. Join point S to P and to R. (Draw $\overline{PS}$ and $\overline{PR}$)

Now Quadrilateral PQRS is the required Rhombus.

8.1.6 Construct a Parallelogram

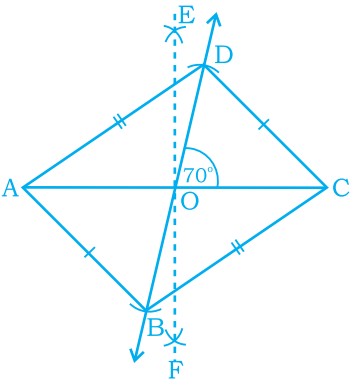
(a) When the two diagonals and angle between them are given.

Example 1. Construct a $\|^m$ ABCD, in which $m\overline{AC} = 6$ cm, $m\overline{BD} = 4.5$ cm and angle between them is of 70°.

Given: In $\|^m$ ABCD, $m\overline{AC} = 6$ cm , $m\overline{BD} = 4.5$ cm and $m\angle COD = 70°$

Required: To construct $\|^m$ ABCD from the given date.

Steps of construction



1. Draw $\overline{AC}$, 6 cm long.
2. Draw $\overleftrightarrow{EF}$ the right bisector of $\overline{AC}$ intersecting $\overline{AC}$ at point O.
3. At point O, with the help of protractor construct $\angle COD$ of measure 70°
4. Take point O as centre and with radius $\frac{4.5}{2} = 2.25$ cm, draw arcs on both sides to cut $\overrightarrow{OD}$ at D and B.

5. Draw $\overline{AB}$, $\overline{AD}$, $\overline{BC}$ and $\overline{CD}$

Now Quad. ABCD is the required parallelogram.

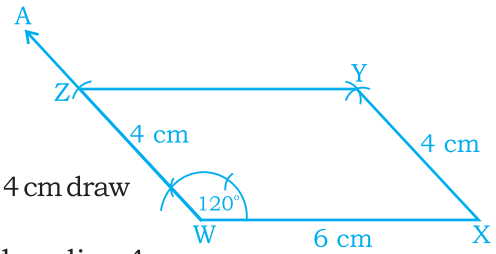
(b) When two adjacent sides and the angle included between them is given.

Example 2. Construct $\|^m$ WXYZ in which $m\,\overline{WX}$ = 6 cm, $m\,\overline{WZ}$ = 4 cm and $m\angle XWZ = 120^\circ$.

Given: In $\|^m$ WXYZ, $m\,\overline{WX}$ = 6 cm, $m\,\overline{WZ}$ = 4 cm and $m\angle XWZ = 120^\circ$

Required: To construct $\|^m$ WXYZ from the given data.

Steps of construction:



1. Draw $\overline{WX}$, 6 cm long.
2. At point W, construct $\angle XWA$ such that $m\angle XWA = 120^\circ$.
3. Take point W as centre, with radius 4 cm draw an arc to cut $\overrightarrow{WA}$ at Z.
4. Take point X as centre and with radius 4 cm draw an arc.
5. Take point Z as centre and with radius 6 cm draw another arc to cut the previous arc at Y.

Now Quad WXYZ is the required parallelogram.

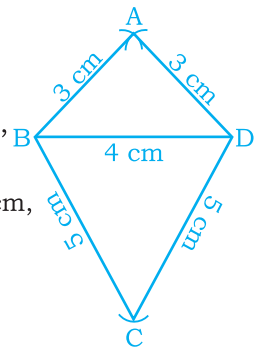
8.1.7 Construct a Kite when two unequal sides and a diagonal are given.

Example 3. Construct Kite ABCD in which $m\,\overline{AB}$ = 3 cm, $m\,\overline{BC}$ = 5 cm and diagonal $\overline{BD}$ = 4 cm

Given: In Kite ABCD, $m\,\overline{AB}$ = 3 cm, $m\,\overline{BC}$ = 5 cm and $m\,\overline{BD}$ = 4 cm

Required: To construct Kite ABCD from the given data.

Steps of construction:



1. Draw $\overline{BD}$, 4 cm long.
2. Take pointB and D as centres and with radius 3 cm, draw arcs to cut one another at point A.
3. Again take points B and D as centres and with radius 5 cm, draw arcs to cut one another at point C.
4. Draw $\overline{AB}$, $\overline{AD}$, $\overline{BC}$ and $\overline{DC}$

Now Quad. ABCD is the required Kite.

8.1.8 Construct a Regular Pentagon when measurement of a side is given.

Example 4. Construct a Regular Pentagon ABCDE, each of whose side measures 4 cm.

Given: In Regular Pentagon ABCDE, $m\overline{AB} = 4$ cm

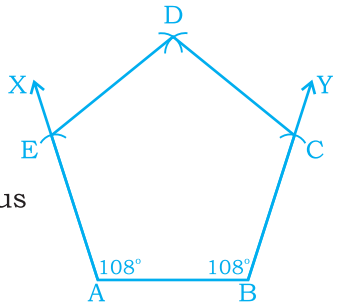
$$m\angle A = m\angle B = \left(\frac{2(5)-4}{5}\right) rt\angle s \quad \text{[from unit-7]}$$

$$= \left(\frac{10-4}{5}\right) \times 90^\circ \quad = \frac{6}{\cancel{5}} \times \overset{18}{\cancel{90}} = 108^\circ$$

Required: To construct Regular Pentagon ABCDE from the given data.

Steps of construction:

1. Draw $\overline{AB}$, 4 cm long.
2. At points A and B, construct, with the help of protractor, $\angle BAX$ and $\angle ABY$ such that each angle measures 108°.
3. Take points A and B as centres, and with radius $m\overline{AB}$, draw arcs to cut $\overrightarrow{AX}$ at E and $\overrightarrow{BY}$ at point C.



4. Take points C and E as centres and with the same radius draw arcs to cut one another at point D.
5. Draw $\overline{CD}$ and $\overline{ED}$.

Now polygon ABCDE is the required Regular Pentagon.

8.1.9 Construct a Regular Hexagon when a side is given.

Example 5.

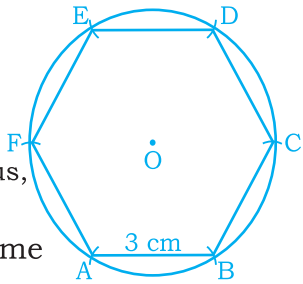
Construct Regular Hexagon ABCDEF each of whose side measures 3 cm.

Given: In Regular Hexagon, $m\overline{AB} = 3$ cm

Required: To construct Regular Hexagon ABCDEF from the given data.

Steps of construction:

1. At a convenient place, take point O as centre and with radius 3 cm construct a circle
2. Take a point A on the circle.
3. Take point A as centre, with radius 3 cm, draw an arc to cut the circle at B.
4. Then taking point B as centre, with the same radius, draw another arc to cut the circle at C.
5. Again take point C as centre and with the same radius, draw an arc to cut the circle at point D and repeat the same process with Centres D and E to cut the circle at points E and F respectively.
6. Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{FA}$.



Now polygon ABCDEF is the required Regular Hexagon.

EXERCISE 8.1

1. Draw similar converging lines in your exercise book as given below, then:

(i) Draw the bisector of the angle formed by the converging lines and (ii) Find the measure of the angle without producing the lines.

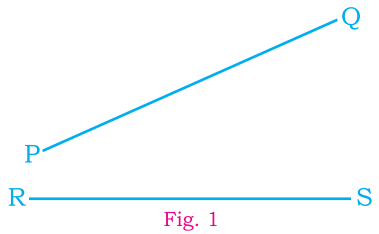


Fig. 1

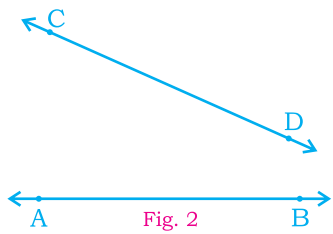


Fig. 2

2. (a) Construct square ABCD in which measure of its diagonal is as under:

(i) $m\overline{AC} = 6$ cm (ii) $m\overline{BD} = 7$ cm (iii) $m\overline{AC} = 8$ cm

(b) Construct square PQRS when the difference between its diagonal and a side is:

(i) 2.5 cm (ii) 3 cm (iii) 3.5 cm (iv) 4.5 cm

(c) Construct square WXYZ when the sum of its diagonal and a side is:

(i) 10 cm (ii) 12 cm (iii) 11 cm (iv) 8 cm

3. (a) Construct rectangle PQRS when its two sides are:

(i) $m\overline{PQ} = 6$ cm, $m\overline{QR} = 4$ cm (ii) $m\overline{PQ} = 4$ cm, $m\overline{QR} = 7$ cm

(b) Construct rectangle ABCD, when its diagonal and one side are given:

(i) $m\overline{AC} = 6$ cm, $m\overline{AB} = 4$ cm (ii) $m\overline{BD} = 7$ cm, $m\overline{AB} = 5$ cm

4. (a) Construct Rhombus ABCD when one side and base angle are given:

(i) $m\overline{AB} = 4$ cm, $m\angle A = 120^\circ$ (ii) $m\overline{AB} = 5$ cm, $m\angle A = 60^\circ$

(b) Construct Rhombus PQRS, when one side and a diagonal are given:

(i) $m\overline{PQ} = 5$ cm, $m\overline{QS} = 4$ cm (ii) $m\overline{PQ} = 4.5$ cm, $m\overline{PR} = 7$ cm

5.(a) Construct $\|^m$ PQRS, when two diagonals and angle between them are given:

(i) $m\overline{QR} = 8$ cm, $m\overline{QS} = 6$ cm, $m\angle QOR = 60^\circ$

(ii) $m\overline{PR} = 6$ cm, $m\overline{QS} = 8$ cm, $m\angle QOR = 120^\circ$

(b) Construct $\|^m$ ABCD, when two adjacent sides and included angle are given:

(i) $m\overline{AB} = 6$ cm, $m\overline{BC} = 4$ cm, $m\angle B = 70^\circ$

(ii) $m\overline{AB} = 4$ cm, $m\overline{AD} = 5$ cm, $m\angle A = 100^\circ$

6. Construct Kite ABCD, when two unequal sides and a diagonal are given:

(i) $m\overline{AB} = 4$ cm, $m\overline{BC} = 7$ cm, $m\overline{AC} = 9$ cm

(ii) $m\overline{AB} = 4$ cm, $m\overline{BC} = 6$ cm, $m\overline{BD} = 4$ cm

7. Construct Regular Pentagon ABCDE each of whose side is:

(i) 3.5 cm (ii) 4 cm (iii) 4.5 cm (iv) 5 cm

8. Construct Regular Hexagon ABCDEF each of whose side is:

(i) 3 cm (ii) 3.5 cm (iii) 2.5 cm (iv) 4 cm

8.2 CONSTRUCTION OF A RIGHT ANGLED TRIANGLE

(a) Construct a right angled triangle when hypotenuse and one side are given:

Example 1. Construct right-angled-Triangle ABC, in which hypotenuse $m\overline{BC} = 10$ cm and $m\overline{AB} = 6$ cm

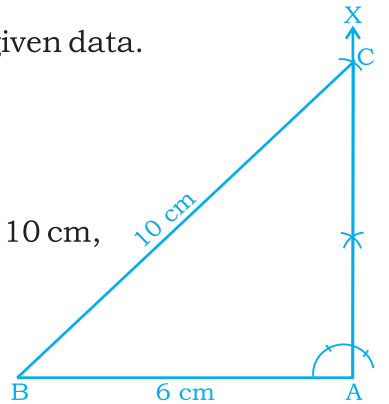
Given: In right angled ΔABC, $m\overline{BC} = 10$ cm, $m\overline{AB} = 6$ cm, $m\angle A = 90^\circ$

Required: To construct ΔABC from the given data.

Steps of construction:

1. Draw $\overline{AB}$, 6cm long.
2. At point A, draw $\overrightarrow{AX} \perp \overline{AB}$.
3. Take point B as centre and with radius 10 cm, draw an arc which cuts $\overrightarrow{AX}$ at point C.
4. Join points B and C.(Draw $\overline{BC}$)

Now ΔABC is the required right triangle.



(b) Construct a right angled triangle when the hypotenuse and the Vertical Height from the vertex to the Hypotenuse are given.

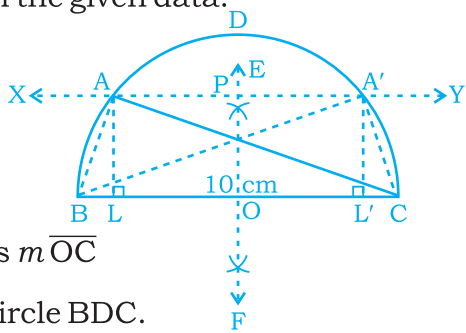
Example 2. Construct a right-angled ΔABC, in which hypotenuse $\overline{BC}$ is 10 cm and vertical height from point A to $\overline{BC}$ is 3.8 cm.

Given: In right angled ΔABC, $m\angle A = 90^\circ$, $m\overline{BC} = 10$ cm, $m\overline{AL} = 3.8$ cm $(\overline{AL} \perp \overline{BC})$

Required: To construct ΔABC from the given data.

Steps of construction:

1. Draw $\overline{BC}$, 10 cm long.
2. Draw $\overleftrightarrow{EF}$ the right bisector of $\overline{BC}$ intersecting it at O.
3. Take O as centre, and with radius $m\overline{OC}$ or $m\overline{OB}\left(\frac{10}{2} = 5 \text{ cm}\right)$ draw semi-circle BDC.



4. Take a point P on $\overrightarrow{OE}$ such that $m\,\overline{OP} = 3.8$cm

5. Through point P, draw $\overleftrightarrow{XY} \parallel \overline{BC}$ (i.e., draw $\overleftrightarrow{XY} \perp \overline{OP}$) cutting the circle at points A and A′.

6. Draw $\overline{AB}$ and $\overline{AC}$ (Also draw $\overline{A'B}$ and $\overline{A'C}$)

Now ΔABC (and $\Delta A'BC$) are the required rt. angled triangles in which

$m\,\overline{AL} = m\,\overline{A'L'} = 3.8$ cm

Note: In any semi-circle, inscribed triangle is a right angled triangle.

EXERCISE 8.2

1. Construct the following right angled triangles when:

(a) Hypotenuse = 10 cm and measure of one side = 8 cm

(b) Hypotenuse = 13 cm and measure of one side = 12 cm

(c) One side $\overline{PQ}$ = 5 cm, one side $\overline{SQ}$ = 6 cm

2. Construct the following right angled triangles when:

(a) Hypotenuse = 10 cm and vertical height from vertex to Hypotenuse = 3 cm

(b) Hypotenuse = 12 cm and vertical height from vertex to Hypotenuse = 4 cm

(c) Vertical height $\overline{PM}$ = 3 cm, Vertical height $\overline{SN}$ = 3.5 cm

(d) Vertical height $\overline{PM}$ = 2.5 cm, Vertical height $\overline{SN}$ = 3 cm

3. Construct the right angled triangles ABC and DBC on either sides of Hypotenuse $\overline{BC}$, 10 cm long when:

(a) Vertical height $\overline{AL}$ = 3.5 cm, $m\,\overline{DC}$ = 5 cm

(b) Vertical height $\overline{AL}$ = 3 cm, $m\,\overline{DB}$ = 6 cm

REVIEW EXERCISE 8

1. Four options are given for each statement. Tick the one which is correct.

(i) Measure of the angle formed by two converging lines (without producing them) is equal to:

(a) Difference of measures of two angles formed by joining the end points of diverging side.

(b) Difference of 180° and the sum of the measures of two angles formed by joining the end-points of diverging side.

(c) 180° – (sum of the measures of angles formed by drawing any transversal any where cutting the two lines).

(d) None of these.

(ii) A polygon with sum of the measures of all interior angles equal to four right angles is _____.

(a) Triangle *(b) Quadrilateral* *(c) Pentagon* *(d) Hexagon*

(iii) A polygon with sum of the measures of all interior angles equal to two right angles is _____.

(a) Triangle *(b) Quadrilateral* *(c) Pentagon* *(d) Hexagon*

(iv) A polygon with sum of measures of all interior angles equal to eight right angles is _____.

(a) Triangle *(b) Quadrilateral* *(c) Pentagon* *(d) Hexagon*

(v) A polygon with sum of measures of all interior angles equal to six right angles is _____.

(a) Triangle *(b) Quadrilateral* *(c) Pentagon* *(d) Hexagon*

(vi) In a rectangle the diagonals _____.

(a) bisect each other at right angles *(b) are perpendicular to each other*
(c) intersect each other at mid-point *(d) None of these*

(vii) Square is a _____.

(a) Rectangle *(b) Regular Polygon*
(c) Quadrilateral having unequal sides *(d) None of these*

(viii) Measure of each interior angle of a Regular Hexagon is _____.

(a) 108° *(b) 120°* *(c) 140°* *(d) None of these*

2. Construct the following:

(i) Square PQRS such that $m\overline{QS} = 6$ cm

(ii) Square WXYZ when the difference of its diagonal and side is 2.5 cm

(iii) Square ABCD when the sum of its diagonal and side is 12 cm.

(iv) Rectangle ABCD, when $m\overline{AB} = 4$ cm, $m\overline{BD} = 6$ cm

(v) Rhombus PQRS, when $m\overline{PQ} = 5$ cm, $m\angle P = 110^\circ$

(vi) Rhombus ABCD, when m $\overline{BC} = 5$ cm, $m\overline{BD} = 7$ cm

(vii) $//^m$ WXYZ whose diagonals are 6 cm and 9 cm and the angle between them is of 70°.

(viii) Kite ABCD when $m\overline{AB} = 6$ cm, $m\overline{BC} = 3$ cm, $m\overline{BD} = 4$ cm.

(ix) Regular Pentagon PQRST when $m\overline{PQ} = 4$ cm.

(x) Regular Hexagon ABCDEF when $m\overline{AB} = 3.5$ cm

3. Draw any two converging line segments $\overline{AB}$ and $\overline{CD}$ and draw the bisector of angle formed by these without producing these lines.

4. Draw any two converging line segments $\overline{PQ}$ and $\overline{PS}$ converging toward points Q and S.

(a) Find the angle between them (to be formed).

(b) Draw the bisector of the angle to be formed without producing the lines.

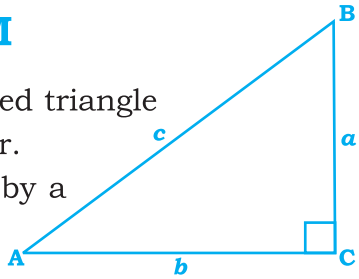
SUMMARY

- Converging lines are non-parallel lines (or line segments) which will intersect if produced.
- Angle between the converging lines without producing them is equal to 180° – (sum of the measures of two angles formed by joining the non-converging end points).
- By joining the In-centre and ex-centre of the triangle (that would be formed if lines are produced) we get the bisector of the angle by converging lines (without producing them) because the bisector of angles are concurrent with these points.

UNIT 9

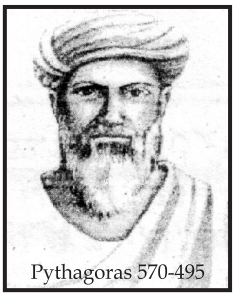
AREAS AND VOLUMES

9.1 PYTHAGORAS THEOREM

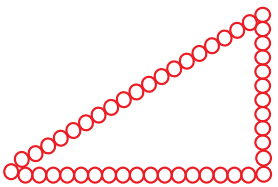


All the three sides of the right angled triangle have a particular relation with one another.

This relation was first discovered by a Greek Mathematician, Pythagoras (570-495 B.C) about 2500 years ago. Hence it is named as Pythagoras Theorem.


Pythagoras 570-495

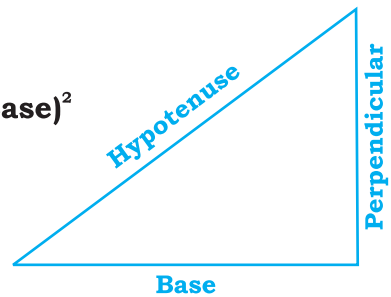
He thought of discovering it when he observed a strangeful method adopted by Egyptians to measure the width of River Nile with the help of a triangle formed by chains. Though there is no other publication of Pythagoras available in any book but because of this one theorem he is well known and famous in the world of science.



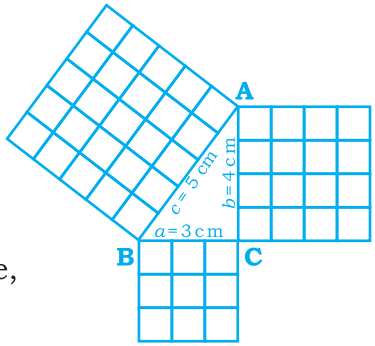
9.1.1 State the Pythagoras Theorem and give its informal proof

In any right angled triangle, the square of hypotenuse is equal to the sum of the squares of two remaining sides i.e. base and altitude (or perpendicular).

(Hypotenuse)^2 = (Perpendicular)^2 + (Base)^2



Informal Proof:



Draw a right angled triangle ABC;

in which $\angle C$ is a right angle.

$m\overline{BC} = 3$ cm $= a$, $m\overline{AC} = 4$ cm $= b$

and $m\overline{AB} = 5$ cm $= c$

According to pythagoras theorem we have,

To prove : $(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2$

Or $c^2 = a^2 + b^2$

Take $\overline{BC} = 3$ cm as one side of a square.

Complete the square. Draw 9 small unit squares in it.

Again take $\overline{AC} = 4$ cm. Complete another square of 4 cm on it. Draw 16 small unit squares in it. Each small square measures 1cm long. Lastly take $\overline{AB} = 5$ cm. Complete the square of sides 5 cm each on $\overline{AB}$. Draw 25 small unit squares on it, such that each small square measures 1cm long.

Thus, the number of small squares of 1 sq cm on Perpendicular plus the number of small squares of 1 sq cm on Base = 16 + 9 = 25, which is the number of small squares of 1 sq cm on the Hypotenuse.

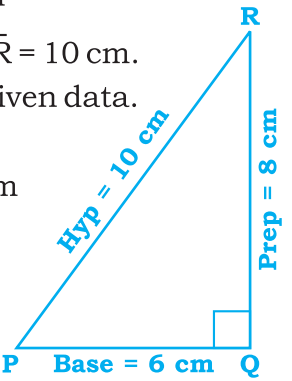
Hence, proved that in any right angled triangle:

(Hypotenuse)^2 = (Perpendicular)^2 + (Base)^2

Let the measure of Hypotenuse, Perpendicular and Base be represented by H, P and B respectively. Then **$H^2 = P^2 + B^2$**

We can prove informally Pythagoras Theorem taking different measures of three sides of a right angled triangle.

Activity: PQR is a right angled triangle in which $m\angle Q = 90^\circ$, $m\overline{PQ} = 6$ cm, $m\overline{QR} = 8$ cm and $m\overline{PR} = 10$ cm. Verify the Pythagoras Theorem with the above given data.



Proof: In given right angled triangle:

The measure of $\overline{QR}$, the Perpendicular is 8cm and therefore $(Prep:)^2 = 8^2 = 64\ cm^2$

The measure of $\overline{PQ}$, the Base is 6cm and therefore $(Base)^2 = 6^2 = 36\ cm^2$

The measure of $\overline{PR}$, the Hypotenuse is 10cm and therefore $(Hyp)^2 = 10^2 = 100\ cm^2$

It is observed that: $100 = 64 + 36$

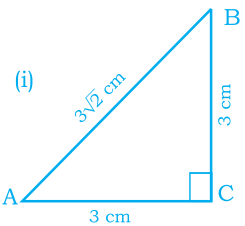
or $(10)^2 = (8)^2 + (6)^2$

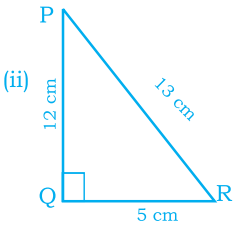
or $(Hyp)^2 = (Prep)^2 + (Base)^2$

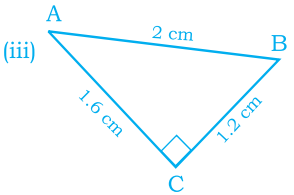
Hence, the Pythagoras Theorem is verified.

EXERCISE 9.1

1. Prove Pythagoras theorem for the following triangles.







2. Prove the Pythagoras Theorem with the help of following Data:

(i) LMN is a right angled triangle in which $m\angle M = 90^\circ$, $m\overline{LN} = 15$ cm, $m\overline{LM} = 9$ cm and $m\overline{MN} = 12$ cm

(ii) PQR is a right angled triangle in which $m\angle Q = 90^\circ$,
$m\overline{PR} = 20$ cm, $m\overline{PQ} = 16$ cm and $m\overline{QR} = 12$ cm

(iii) XYZ is a right angled triangle in which $m\angle Y = 90^\circ$,
$m\overline{XZ} = 25$ cm, $m\overline{XY} = 15$ cm and $m\overline{YZ} = 20$ cm

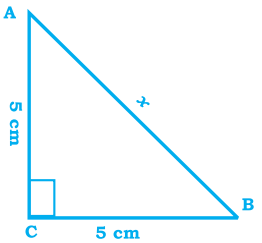
9.1.2 Solve right angled triangles using Pythagoras Theorem

We can find the measure of any one of the three sides, i.e. Hypotenuse, Perpendicular or Base , if the measures of two of them are given in the following way:

(i) $H^2 = P^2 + B^2$ (ii) $B^2 = H^2 - P^2$ (iii) $P^2 = H^2 - B^2$

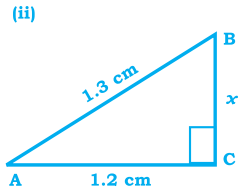
Example 1. Find the unknown side in right angled triangle ABC.

(i)



Solution: Here $m\overline{AC} = 5$ cm
$m\overline{BC} = 5$ cm, $m\overline{AB} = ?$
By Pythagoras Theorem
$H^2 = P^2 + B^2$
or $(m\overline{AB})^2 = (m\overline{AC})^2 + (m\overline{BC})^2$
I.e. $x^2 = (5)^2 + (5)^2 = 25+25=50$
$\Rightarrow x^2 = 50 \Rightarrow x = \sqrt{50} = \sqrt{5\times5\times2}$
$\Rightarrow x = 5\sqrt{2}$ cm.

(ii)



Solution: Here $m\overline{AC} = 1.2$ cm
$m\overline{BC} = x = ?$, $m\overline{AB} = 1.3$ cm
By Pythagoras Theorem
$H^2 = P^2 + B^2$
or $(m\overline{AB})^2 = (m\overline{AC})^2 + (m\overline{BC})^2$
i.e. $(1.3)^2 = (1.2)^2 + x^2$
$\Rightarrow 1.69 = 1.44 + x^2$
$\Rightarrow 1.69 - 1.44 = x^2 \Rightarrow x^2 = 0.25$
$\Rightarrow x = \sqrt{0.5\times0.5} = 0.5$ cm.

Example 2. The measure of Perpendicular of right angled triangle is 60 cm and its base is 45 cm. Find the measurement of the Hypotenuse.

Solution: With the help of Pythagoras Theorem, we can find:

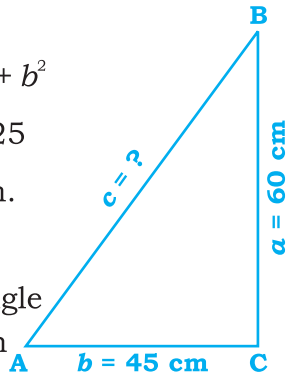
$(\text{Hyp})^2 = (\text{Perp})^2 + (\text{Base})^2$

or $(m\overline{AB})^2 = (m\overline{BC})^2 + (m\overline{AC})^2$ or $c^2 = a^2 + b^2$

$\Rightarrow c^2 = (60)^2 + (45)^2 = 3600 + 2025 = 5625$

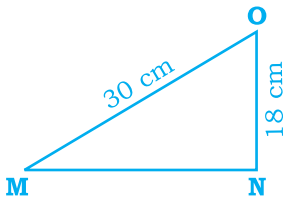
$\Rightarrow c = \sqrt{5625} = 75$ Therefore $c = 75$ cm.

Hence, the measure of Hypotenuse is 75 cm.



Remember: To solve a right angled triangle means to find the measures of all its sides with the help of Pythagoras theorem.

Example 3. Solve a right angled triangle MNO in which $m\angle N = 90^\circ$, $m\overline{MO} = 30$ cm and $m\overline{NO} = 18$ cm



Solution: Pythagoras Theorem states that:

$(\text{Hyp})^2 = (\text{Perp})^2 + (\text{Base})^2$

From the figure, it is clear that $m\overline{MO} = 30$ cm = Hyp.
and $m\overline{NO} = 18$ cm = Perp.

In order to find the measure of base of the given right angled triangle, we must deduce from Pythagoras Theorem.

$(\text{Base})^2 = (\text{Hyp})^2 - (\text{Perp})^2$

$= (30)^2 - (18)^2$

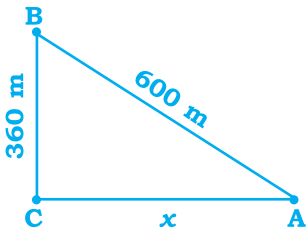
$= 900 - 324 = 576$

or Base $= \sqrt{576} = \sqrt{(24)^2} = 24$ cm.

Hence, Base $= 24$ cm.

Example 4. An old woman takes a short cut from her home to Tower along a footpath AB. How much further would it be for the old woman to the Market Tower by going from A to C and then from C to B?

Solution: When the old woman takes a shortcut ($\overline{AB}$) from her home at A to the Market Tower B; she completes a distance of 600 m. Again if she takes an alternative route, the distance she walk which is:



$$m\overline{AC} + m\overline{CB} = (x + 360)\text{ m}$$

Here ACB is a right angled triangle with the Hypotenuse

$$m\overline{AB} = 600\text{ m}$$

Therefore $(\text{Hyp})^2 = (\text{Perp})^2 + (\text{Base})^2$, by Pythagoras Theorem.

$$(600)^2 = (360)^2 + (\text{Base})^2$$

or $$360000 = 129600 + x^2$$

Therefore $$x^2 = 360000 - 129600 = 230400$$

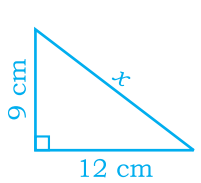
or $$x = \sqrt{230400} = 480\text{ m}$$

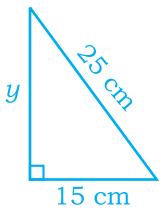
	480
4	$\overline{23}\,\overline{04}\,\overline{00}$
+4	−16
88	704
+8	−704
960	000
+0	−000
960	0

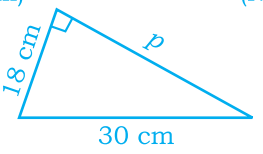
Total length of the alternative route is 360 + 480 = 840 m.
Thus the extra distance = (840 – 600) m = 240 m.

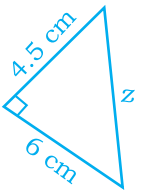
EXERCISE 9.2

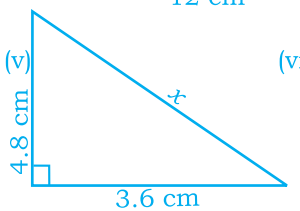
1. For each of the following right angled triangles, calculate the measure of unknown side.

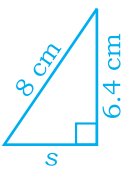
(i)


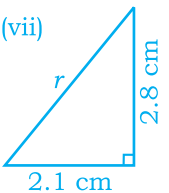
(ii)


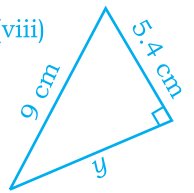
(iii)


(iv)


(v)


(vi)


(vii)


(viii)


2. Let l, m and n be the lengths of the sides of a triangle LMN.

Then decide which of the following triangles are not right angled triangles. When any of $\angle L$, $\angle M$ and $\angle LNM$ may be a right angle.

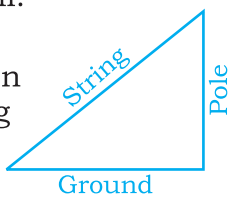
(i) $l = 9$, $m = 12$ and $n = 15$ (ii) $l = 7$, $m = 8$ and $n = 9$
(iii) $l = 6$, $m = 8$ and $n = 10$ (iv) $l = 12$, $m = 16$ and $n = 20$
(v) $l = 2$, $m = 4$ and $n = 6$ (vi) $l = 1.5$, $m = 2$ and $n = 2.5$

3. Following are isosceles right angled triangles. The square of Hypotenuse is given. Find the length of congruent sides.

(i) $H^2 = 50\ cm^2$ (ii) $H^2 = 72\ cm^2$ (iii) $H^2 = 98\ cm^2$
(iv) $H^2 = 128\ cm^2$ (v) $H^2 = 2.42\ cm^2$ (vi) $H^2 = 9.68\ cm^2$
(vii) $H^2 = 4.5\ cm^2$ (viii) $H^2 = 60.5\ cm^2$ (ix) $H^2 = 112.5\ cm^2$

4. A ladder 6.5m long is placed against a wall. If its upper end reaches the height of 5.2 m along the wall, then find the distance of the foot of the ladder from the wall.

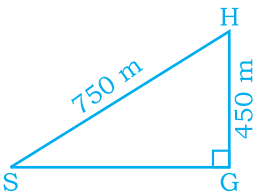
5. An electrical pole of 12 m height is stretched by a string of 20 m. Find the distance between the foot of the pole and the end point of string on the ground.



6. Find the length of diagonals in each of the following rectangles when their lengths and widths are given as under:

(i) Length = 6 cm and Width = 6 cm
(ii) Length = 44 cm and Width = 33 cm
(iii) Length = 5.6 cm and Width = 4.2 cm
(iv) Length = 6.4 cm and Width = 4.8 cm

7. A student takes a short cut from his home (H) to the school (S) along a footpath (HS). How much further would it be for the student to walk to the school by going from House (H) to the Garden (G) and then from Garden to School (S)?



8. Find the area of a rectangular field whose width is 48 m and the length of diagonal is 80 m.

9. Consider the figures and find the value of unknown measures of x and y.

Fig. (i)

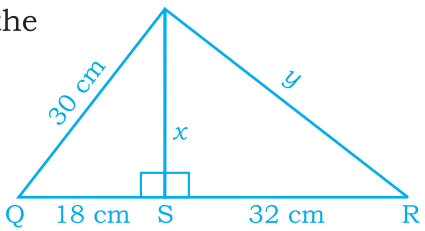


Fig. (ii)

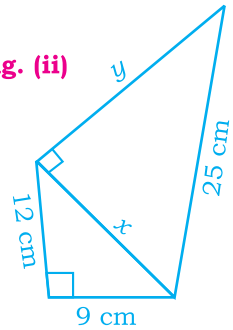
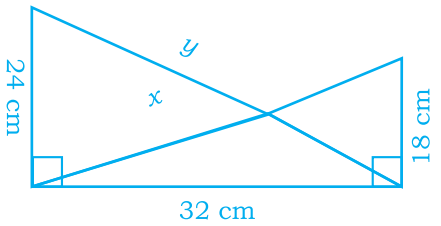
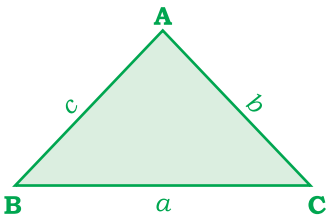


Fig. (iii)



9.2 HERO'S FORMULA

Area of a triangular region is the space covered by the three sides of a triangle. Here shaded portion along with the sides of the triangle is the triangular region ABC. There are many methods to find the area of a triangular region. In previous classes, we have learnt how to find the area of a triangular region when its base and altitude are given by the formula:

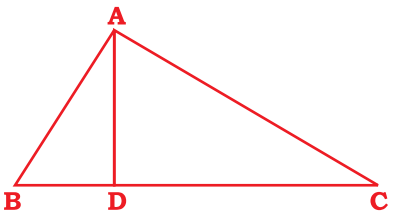


Area of triangular region = $\triangle$ ABC

$= \frac{1}{2}$ base $\times$ altitude

$= \frac{1}{2}\,(m\overline{BC}) \times (m\overline{AD}) = \frac{1}{2}\,(x \times y)$

where $m\overline{BC} = x$ and $m\overline{AD} = y$.



Here we shall learn to find the area of a triangular region when the measures of three sides of a triangle are given.

Let a, b, c denote the lengths of sides of a triangle ABC, then:

Area of triangle ABC = $\triangle ABC = \sqrt{S(S-a)(S-b)(S-c)}$

where S = semi perimeter of ΔABC. This is known as Hero's Formula.

Therefore $S = \frac{1}{2}$ (sum of measures of three sides) $= \frac{1}{2}(a+b+c)$

or $S = \frac{(a+b+c)}{2}$

This formula was introduced by a "Greek Mathematician Hero of Alexandria" and is named as "Hero's Formula". The above mentioned formula is very easy to understand. With the help of this formula we shall be able to get the area of triangular plots, triangular rooms, triangular sheets, courtyards or fields. We will not seek help of any tapedar etc for the measurement of agricultural lands particularly in rural areas.

9.2.1 State and apply Hero's Formula to find the area of a triangular and quadrilateral regions.

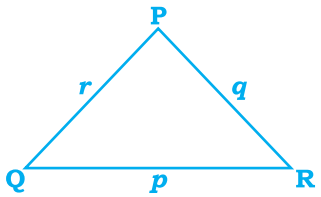
(A). State and apply Hero's Formula to find the area of a triangular region.

Statement of Hero's Formula:

If p, q, r are the lengths of a triangle PQR, then area of the triangle PQR denoted as $\triangle$, is given by:

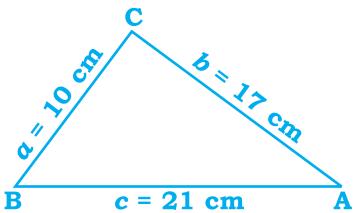
$$\triangle PQR = \sqrt{S(S-p)(S-q)(S-r)}$$

where $S = \frac{p+q+r}{2}$



Example 1. Using Hero's Formula, find the area of triangle ABC when the lengths of its sides are 10 cm, 17 cm and 21 cm.

Solution: Look at the figure. Here a denotes the side opposite to $\angle A$, b the side opposite to $\angle B$ and c denotes the side opposite to $\angle C$. We shall first find



$$S = \frac{a+b+c}{2} = \frac{10+17+21}{2} = \frac{48}{2} = 24 \text{ cm}$$

Therefore S = 24 cm

Putting the values of a, b, c and S in the formula

$$\text{Area of } \Delta ABC = \triangle ABC = \sqrt{S(S-a)(S-b)(S-c)}$$

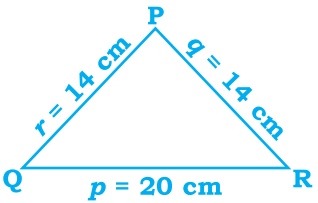
$$\triangle ABC = \sqrt{24 \times (24-10) \times (24-17) \times (24-21)}$$

$$= \sqrt{24 \times 14 \times 7 \times 3} = \sqrt{2 \times 2 \times 2 \times 3 \times 2 \times 7 \times 7 \times 3}$$

$$= \sqrt{\underline{2 \times 2} \times \underline{2 \times 2} \times \underline{3 \times 3} \times \underline{7 \times 7}} = 2 \times 2 \times 3 \times 7 = 84$$

Hence, the area of the given ΔABC is 84 cm^2.

Example 2. Using Hero's Formula, find the area of an isosceles triangle PQR in which $m\overline{PQ} = m\overline{PR} = 14$ cm and $m\overline{QR} = 20$ cm.



Solution: Let p, q and r be the sides opposite to the angles P, Q and R respectively.

Then $m\overline{PQ} = r = 14$ cm, $m\overline{PR} = q = 14$ cm and $m\overline{QR} = p = 20$ cm

Now $S = \frac{p+q+r}{2} = \frac{20+14+14}{2} = \frac{48}{2} = 24$ cm

$$\triangle PQR = \sqrt{S(S-p)(S-q)(S-r)}$$

$$= \sqrt{24(24-20)(24-14)(24-14)}$$

$$= \sqrt{24 \times 4 \times 10 \times 10} = \sqrt{6 \times \underline{4 \times 4} \times \underline{10 \times 10}}$$

$$= 4 \times 10\sqrt{6} = 40\sqrt{6} \text{ cm}^2$$

$$= 40 \times (2.45) = 98 \text{ cm}^2$$

Hence, area of the given isosceles triangle is 98 cm^2.

Example 3. Using the Hero's formula, find the missing elements:

$a = 10$ cm, $b = 12$ cm, $S = 18$ cm, $c =$ _____, $\triangle ABC =$ _____

Solution: Hero's Formula states:

$$\triangle ABC = \sqrt{S(S-a)(S-b)(S-c)} \quad \text{where } S = \frac{a+b+c}{2}$$

From $S = \frac{a+b+c}{2}$ we can find the value of the third side 'C'

Putting the values of S, a and b, we get

Since $18 = \frac{10+12+c}{2}$, Therefore by cross-multiplication,

we get $36 = 22 + c$

or $c = 36 - 22 = 14$ cm

Now $\triangle ABC = \sqrt{S(S-a)(S-b)(S-c)}$

Putting the values of S, a, b, c we get:

$$ABC = \sqrt{18 \times (18-10) \times (18-12) \times (18-14)}$$

$$\triangle = \sqrt{18 \times 8 \times 6 \times 4}$$

$$= \sqrt{\underline{3 \times 3} \times \underline{2 \times 2} \times \underline{2 \times 2} \times 2 \times 3 \times \underline{2 \times 2}}$$

$$= 3 \times 2 \times 2 \times 2 \sqrt{6} = 24 \sqrt{6} \text{ cm}^2 = 24 \times 2.45 \text{ cm}^2$$

$$= 58.80 \text{ cm}^2$$

Thus, the missing elements are:

Side $c = 14$ cm and $\triangle ABC = 58.8$ cm^2.

EXERCISE 9.3

1. Using Hero's Formula, find the area of the following triangles; when the length of their sides is given as under:

(i) Triangle ABC in which $a = 16$ cm, $b = 20$ cm and $c = 16$ cm

(ii) Triangle DEF in which $d = 10$ cm, $e = 17$ cm and $f = 21$cm

(iii) Triangle LMN in which $l = 6$ cm, $m = 8$ cm and $n = 10$ cm

(iv) Triangle PQR in which $p = 20$ cm, $q = 24$ cm and $r = 18$ cm

(v) Triangle XYZ in which $x = 15$ cm, $y = 21$ cm and $z = 14$ cm

2. Using Hero's Formula, find the area of the following issceles triangles; when the length of their sides is given as under:

(i) Triangle ABC in which $a = 10$ cm, $b = 10$ cm and $c = 12$ cm
(ii) Triangle DCE in which $d = 12$ cm, $c = 12$ cm and $e = 16$ cm
(iii) Triangle LMN in which $l = 9$ cm, $m = 9$cm and $n = 12$ cm
(iv) Triangle PQR in which $p = 24$ cm, $q = 24$ cm and $r = 40$ cm

3. Using Hero's formula find the area of the equilateral triangle with the length of the side is 'a' cm each.

4. Find the length of the missing side of the triangle. Also find the area of the triangle with the help of Hero's Formula.

(i) a = 13 cm, b = 15 cm, S = 20 cm, c = ____ and △ ABC =____

(ii) d = 3.6 cm, e = 4.5 cm, S = 6.0 cm, f = _____ and △ DEF =____

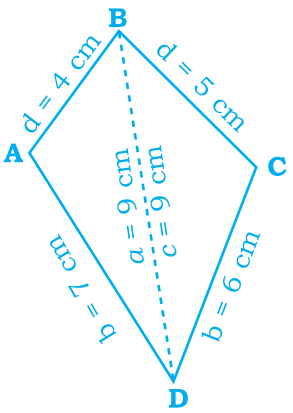
(iii) l = 7.2 cm, m = 6.4 cm, S = 10.0 cm, n = _____ and △ LMN =____

(iv) p = 4.5 cm, q = 4.5 cm, S = 7.0 cm, r = _____ and △ PQR =____

(B). State and apply Hero's Formula to find the area of a quadrilateral region.

We have already learnt to find the area of triangular regions by using Hero's Formula in previous topic.
Now we shall learn to find area of quadrilateral regions with help of Hero's Formula.
Let us see the following example.

Example 1. Find the area of given quadrilateral ABCD by using Hero's Formula.



Solution: Let us draw the figure of quadrilateral ABCD. Here two opposite vertices B and D are joined together. In this way we get, two triangles ΔABD and ΔBCD. Now we can find the area of these two triangles ΔABD and ΔBCD with the help of Hero's Formula as follows:

(i) For $\triangle$ ABD: $a = 9$ cm, $b = 7$ cm, $d = 4$ cm

$$S_1 = \frac{a + b + d}{2} = \frac{9 + 7 + 4}{2} = \frac{20}{2} = 10$$

By Hero's Formula

$$\triangle ABD = \sqrt{S\ (S-a)\ (S-b)\ (S-d)}$$

$$= \sqrt{10 \times (10-9) \times (10-7) \times (10-4)}$$

$$= \sqrt{10 \times 1 \times 3 \times 6} = \sqrt{5 \times 2 \times 1 \times 3 \times 3 \times 2}$$

$$= 2 \times 3\sqrt{5} = 6\sqrt{5}$$

$$= 6 \times 2.4 = 13.44 \text{ cm}^2$$

(ii) $\triangle$ BCD: $b = 6$ cm, $c = 9$ cm, $d = 5$ cm

$$S_2 = \frac{6 + 9 + 5}{2} = \frac{20}{2} = 10 \text{ cm}$$

By the Hero's Formula

$$\triangle BCD = \sqrt{S_2\ (S_2 - b)\ (S_2 - c)\ (S_2 - d)}$$

$$= \sqrt{10 \times (10-6) \times (10-9) \times (10-5)}$$

$$= \sqrt{10 \times 4 \times 1 \times 5}$$

$$= \sqrt{2 \times 5 \times 2 \times 2 \times 5}$$

$$= \sqrt{2 \times \underline{5 \times 5} \times \underline{2 \times 2}}$$

$$= 5 \times 2\sqrt{2} = 10\sqrt{2}$$

$$= 10 \times 1.41 \text{ (Rounded upto two places of decimal)}$$

$$= 14.1 \text{ cm}^2$$

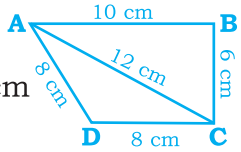
On adding the two results we get the required area of the given quadrilateral ABCD.

Thus, total area of quadrilateral ABCD = $\triangle$ABD + $\triangle$BCD

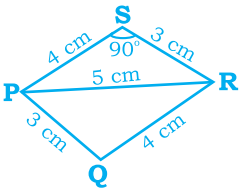
$$= 13.44 + 14.1 = 27.54 \text{ cm}^2$$

EXERCISE 9.4

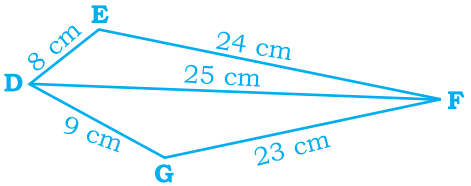
1. Find the area of quadrilateral region ABCD when $m\overline{AB}$ = 10cm, $m\overline{CD}$ = 8cm, $m\overline{AC}$ = 12 cm $m\overline{AD}$ = 8 cm, and $m\overline{BC}$ = 6 cm



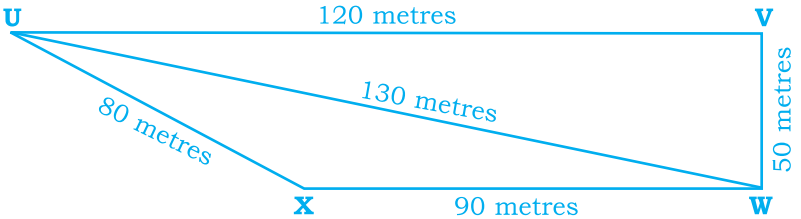
2. Find the area of quadrilateral PQRS when $m\overline{PS}$ = $m\overline{QR}$ = 4 cm, $m\overline{PR}$ = 5 cm and $m\overline{PQ}$ = $m\overline{SR}$ = 3 cm



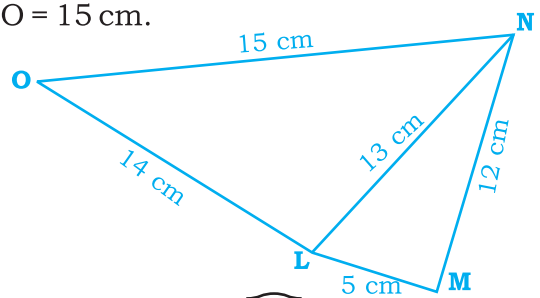
3. Find the area of quadrilateral region DEFG when $m\overline{DE}$ = 8 cm, $m\overline{EF}$ = 24 cm $m\overline{DF}$ = 25 cm, $m\overline{DG}$ = 9 cm and $m\overline{GF}$ = 23 cm



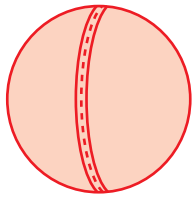
4. Find the area of a quadrilateral field whose measurements of sides are mentioned below:



5. Find the area of quadrilateral region LMNO, when $m\overline{LM}$ =5 cm, $m\overline{MN}$ = 12 cm, $m\overline{LN}$ = 13 cm, $m\overline{LO}$ = 14 cm and $m\overline{NO}$ = 15 cm.

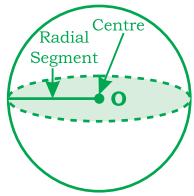


9.3 SURFACE AREA AND VOLUME OF A SPHERE



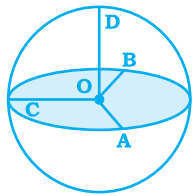
SPHERE:

A cricket ball, a hockey ball, a volley-ball, a football, an iron bob are the examples of a sphere.



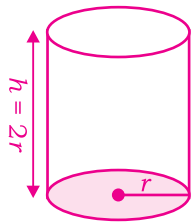
Thus, sphere is a solid thing or material bounded by a single curved surface and is such that all the points on its outer surface are at an equal distance from a fixed point called its centre which is inside the sphere.

In the given figure of sphere, point O is its centre.



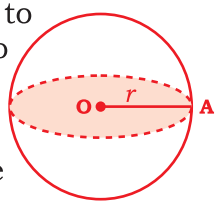
The distance from centre to its outer surface is called its radius. In the figure $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$ are its radial sagments and $m\overline{OA} = m\overline{OB} = m\overline{OC} = m\overline{OD}$, as A, B, C, D lie on the outer surface.

9.3.1 Find the surface area and volume of a sphere.



(A) Surface Area of a Sphere

A famous scientist Archimedes discovered that the surface area of a sphere is equal to the curved surface area of the cylinder whose radius is equal to the radius of the sphere and its height is equal to the diameter of the sphere i.e., $h = 2r$.



Consider the figure, let the radius of sphere = r

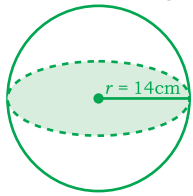
Radius of cylinder is also equal to r. Height of the cylinder is $h = 2r$

We have already learnt in previous class that area of curved surface of cylinder = $2\pi rh$.

Now surface area of sphere = $2\pi r(2r) = 4\pi r^2$ $(\because h = 2r)$

Thus, surface area of the sphere = $4\pi r^2$

Example 1.
Find the surface area of sphere when its radius =14cm and $\pi = \frac{22}{7}$



Solution: Surface area of sphere = $4\pi r^2$

$$= 4 \times \frac{22}{7} \times (14)^2 = 4 \times \frac{22}{\cancel{7}_1} \times \overset{2}{\cancel{14}} \times 14$$

$$= 8 \times 22 \times 14 = 176 \times 14 = 2464 \text{ cm}^2$$

Thus, the surface area of the given sphere is 2464 cm^2.

Example 2. Find the radius of a sphere if the surface area of sphere is 5544 cm^2.

Solution: Surface area of the sphere = $4\pi r^2 = S$, or $r^2 = \frac{S}{4\pi}$

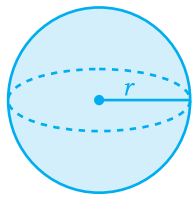
$$\text{Or} \quad r = \sqrt{\frac{S}{4\pi}}$$

$$\text{Radius of sphere} = r = \sqrt{\frac{5544 \times 7}{4 \times 22}} = \sqrt{\frac{\cancel{693}\ 63}{\cancel{1386}\ \cancel{5544} \times 7}{\underset{1}{\cancel{4}} \times \underset{1}{\cancel{22}}}}$$

$$r = \sqrt{63 \times 7} \Rightarrow r = \sqrt{\underline{3 \times 3} \times \underline{7 \times 7}} \Rightarrow r = 3 \times 7 = 21\text{cm}$$

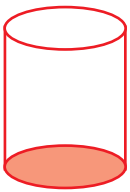
Thus, the radius of the given sphere is 21cm.

(B) Volume of a Sphere



Consider a sphere of radius r and a cylinder of radius r and height $h = 2r$. After so many experiments, it is observed that Volume of a sphere V = Two-third of the volume of cylinder

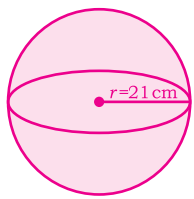
$$V = \frac{2}{3} \times \pi r^2 h = \frac{2}{3} \pi r^2 (2r) = \frac{4}{3} \pi r^3$$



Hence, Volume of the sphere $V = \frac{4}{3} \pi r^3$

Example1. Find volume of sphere if its radius is:
(i) r =7cm **(ii)** r =2.1 m

Solution (i): Volume of sphere $= \frac{4}{3} \pi r^3$.



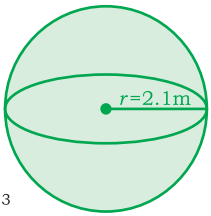
$$= \frac{4}{3} \times \frac{22}{7} \times (7)^3 = \frac{4 \times 22 \times \overset{1}{\cancel{7}} \times 7 \times 7}{3 \times \cancel{7}} =$$

$$\left(\frac{4}{3} \times 22 \times 1 \times 7 \times 7\right) \text{cm}^3 = \frac{4312}{3} \overset{1}{\text{cm}^3} = 1437.33 \text{cm}^3$$

Thus, the Volume of the sphere is 1437.33 cm^3.

Solution (ii) : Volume of a spherical tank $= \frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \frac{22}{7} \times (2.1)^3 m^3 = \frac{4}{3} \times \frac{22}{7} \times \left(\frac{21}{10}\right)^3$

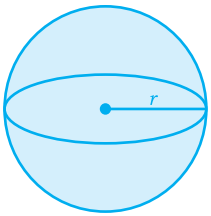


$= \frac{4}{\cancel{3}_1} \times \frac{22}{\cancel{7}} \times \frac{\cancel{21}^{1\cancel{7}}}{10} \times \frac{21}{10} \times \frac{21}{10}$

$= \frac{4 \times 22 \times 21 \times 21}{10 \times 10 \times 10} = \frac{38808}{1000} m^3 = 38.808\ m^3$

Therefore, the volume of given sphere is $38.808\ m^3$

Example 2. Find the volume of a sphere whose surface area is $5544\ cm^2$.



Solution: Let r be the radius of the sphere.

Therefore $A = 4\pi r^2 = 5544$

or $r^2 = \frac{5544}{4\pi} = \frac{5544}{4 \times \frac{22}{7}} = \frac{5544 \times 7}{4 \times 22}$

or $r^2 = \frac{\cancel{5544}^{63\ \cancel{693}\ \cancel{1386}} \times 7}{\cancel{4}_1 \times \cancel{22}_{\cancel{11}_1}} = 63 \times 7 = 3 \times 3 \times 7 \times 7 = 3^2 \times 7^2$

Therefore $r = \sqrt{3^2 \times 7^2} = 3 \times 7 = 21\ cm = \frac{21}{100} m = 0.21 m$

We know that the volume of a sphere $= V = \frac{4}{3}\pi r^3$

$\therefore \quad V = \frac{4}{3}\pi r^3 = \left[\frac{4}{3} \times \frac{22}{7} \times (0.21)^3\right]$

or $V = \left[\frac{4 \times 22 \times \cancel{0.21}^{0.1\ \cancel{0.7}} \times 0.21 \times 0.21}{\cancel{3} \times \cancel{7}_1}\right]$

or $V = (88 \times 0.1 \times 0.21 \times 0.21) m^3 = 0.038808\ m^3$

Note: We know $1 m^3 = 1000$ litres

Therefore, capacity (volume) of the given sphere $= 0.038808\ m^3$

$= (0.038808 \times 1000)$ litres $= 38.808$ litres

EXERCISE 9.5

1. Find the surface area of the following spheres whose radii are given as under:
(i) 35 cm (ii) 56 cm (iii) 0.42 cm (iv) 0.63 cm (v) 0.98 cm

2. Find the radius of each of the following spheres whose surface area are as under?
(i) 616 cm^2 (ii) 2464 cm^2 (iii) 88,704 cm^2 (iv) 985.6 m^2 (v) 154 m^2

3. Find the volume of the following spheres whose radii are:
(i) 21cm (ii) 49 cm (iii) 2.8 m (iv) 77 cm (v) 3.5 m

4. How many litres of water a sphere can contain whose radius is given as under:
(i) 70 cm (ii) 105 cm (iii) 2.8 m (iv) 175 cm (v) 2.45 m

5. Fnd the volume and capacity in litres of a sphere whose area is given as under?
(i) 2464 cm^2 (ii) 9856 cm^2 (iii) 29.8144 m^2
(iv) 43.12 m^2 (v) 13.86 m^2

6. If radius of a sphere becomes double, determine the change in its: (a) Surface Area (b) Volume

7. The radius of sphere A is thrice that of a sphere B. Find:
(i) The ratio among their surface areas.
(ii) The ratio among their volumes.

8. The surface area of a sphere of iron is 77000 cm^2. What will be its volume, if it is melted? How many spheres of radius 1cm can be made out of it?

9. A solid zinc sphere of radius 7cm is melted and a wire of diameter 0.7 cm is made out of the zinc obtained. Find the length of the wire.

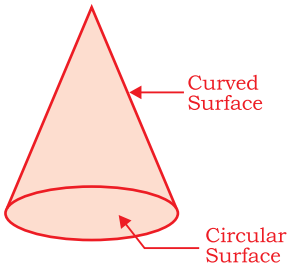
10. Find the volume of a football having radius 12 cm.

9.3.2 Find the surface and volume of a cone

Definition: The given figure is of a CONE.

Let us observe carefully the figure of cone.

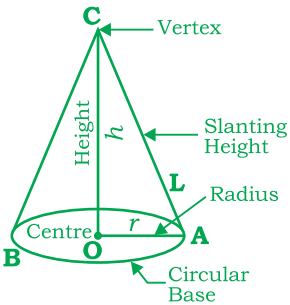
A solid cone consists of two parts:



(i) Curved Surface: It starts from a vertex and ends to a circular region.

(ii) Circular Surface: It is base of a cone. It is circular surface region.

Thus there are five different elements of a cone as shown in the following figure.



1. The Vertex of the Cone is point C.

2. Centre of Circular Base of the Cone is point O.

3. Radius of Circular Base of Cone is $m\overline{OA}$.

4. Height: It is the measure of the line segment joining the vertex C to the centre of the cone O($\overline{CO}$). It is also perpendicular to the radial segment of the cone.

5. Slanting Height of the Cone is $m\overline{AC}$ or $m\overline{BC}$.

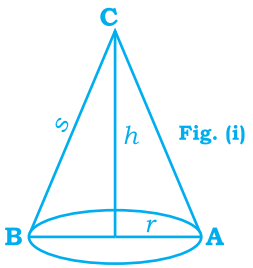
(A) To Find The Surface Area of A Cone

Let us derive a formula for finding the surface area of a cone.

The surface area of a cone consists of area of circular region and also the area of the curved surface of the cone, i.e.

Surface area of a cone = Area of circular region + Area of curved surface.

We have already learnt that area of a circular region = πr^2, where r is its radius.



Now we find the area of the curved surface.

Suppose the height and slanting height of the cone are h and s, respectively.

If we cut a cone from a vertex C in straight to the point B with pair of scissors and then make it flat; we get a sector CBB′.

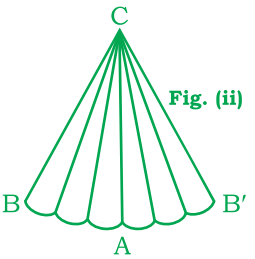


Fig. (ii)

Again if we cut a cone from a Vertex C straight to the point B with a pair of scissors and then make it flat. We get a sector CBAB′ with centre at C and radius equal to slanting heights as shown in Fig (ii).

Now if the arc of the sector is divided into smaller parts and then the same are joined with point C, we see that area of the sector will have many smaller triangular regions which are cut and arranged with their radii up and down. Thus, we get a rectangular shape as shown in Fig. (iii).

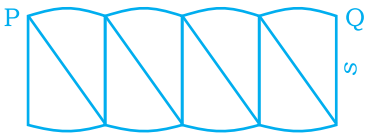


Fig. (iii)

Now let us observe these figures and then write the results as follows:

In Fig. (i) Circumference of the circular base of the cone = $2\pi r$

In Fig. (ii) Length of the arc of the sector = $2\pi r$

In Fig. (iii) Length of the side of rectangular region is $m\overline{PQ} = \frac{1}{2}(2\pi r) = \pi r$

Breadth of rectangle = s

Area of rectangle = Length × Breadth

= $m\overline{PQ} \times s$

= $\pi r \times s = \pi rs$

Hence, area of the curved surface of a cone

= (Length of its slanting surface) × (radius of base) × $\pi = \pi rs$

Area of its circular region = πr^2

Hence, the total surface area of the cone = Area of the curved Surface of the cone + Area of the circular region.

Thus, total surface area of the cone = $\pi rs + \pi r^2 = \pi r(s + r)$

Total surface area of a cone = $\pi r(r + s)$

With the symbols having the meaning as described above.

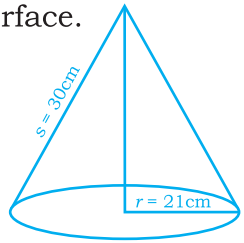
Example 1. The radius of circular base of a cone is 21cm and its slanting height is 30 cm. Find area of its curved surface.

Solution: Radius of base of the cone $= r = 21$ cm

Slanting height of cone $= s = 30$ cm

Area of curved surface $= \pi rs = \frac{22}{\cancel{7}_1} \times \overset{3}{\cancel{21}} \times 30$

$= 22 \times 3 \times 30 = 1980 \text{ cm}^2$



Example 2. The radius of the base of a cone is 15 cm and slanting height is 20 cm . Find the total surface area of the cone.

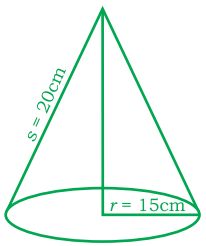
Solution: Radius of base of the cone $= r = 15$ cm

Slanting height $= s = 20$ cm

Total surface area $= \pi r(r + s) = \frac{22}{7} \times 15 \ (15 + 20)$

$= \frac{22 \times 15}{\cancel{7}_1} \times \overset{5}{\cancel{35}}$

$= 22 \times 75 = 1650 \text{ cm}^2$



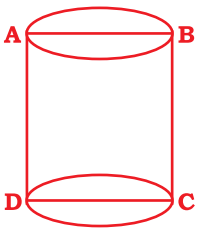
(B) To Find The Volume Of A Cone

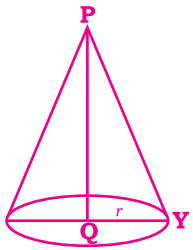
To derive a formula for finding volume of a cone through an activity.

Activity: Take an empty cylinder with one side open and a cone.

The base of cone and base of cylinder are equal and perpendicular length (height) of both should be same.

Look at both the figures of a cone and a cylinder.





It is observed that $m\overline{CD} = m\overline{XY} = 2r$ where r represents the radius of circular surface of the cylinder and the cone and h represents the height of both. Now fill the cone with sand or water and pour it in the cylinder.

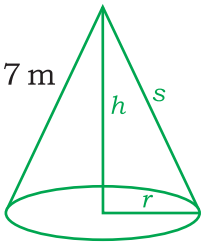
Repeat this process second and third time and observe that cylinder is fully filled with water or sand. From this it is concluded that the quantity of water or sand in the cylinder is thrice that in the cone. We can say that volume of the cone is one third the volume of the cylinder, if their radial segments and heights are equal.

Hence, volume of a cone $= \frac{1}{3}$ (volume of a cylinder), having equal heights and the radii of their bases.

We know that volume of cylinder $= \pi r^2 h$

Hence, volume of a the cone $= \frac{1}{3} \pi r^2 h$

Example 1. The radius of a cone is 2.5 m and its height is 3.5 m Find the volume of the cone.



Solution: Radius of circular base of the cone = r = 2.7 m

Height of the cone = h = 3.5 m

Now Volume of the cone $= \frac{1}{3} \pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times (2.7)^2 \times (3.5)$$

$$= \frac{1 \times 22 \times \overset{0.9}{\cancel{2.7}} \times 2.7 \times \overset{0.5}{\cancel{3.5}}}{{}_1\cancel{3} \times \cancel{7}_1}$$

$$= 22 \times 0.9 \times 2.7 \times 0.5 = 26.73 \text{ m}^3$$

Hence, volume of given cone is 26.73 m^3.

Example 2. The area of the base of cone is 154 cm^2. If its height is thrice the radius of the base, find its volume.

Solution: The formula for finding base area of the cone is $A = \pi r^2$. Here base area given is 154 cm^2.

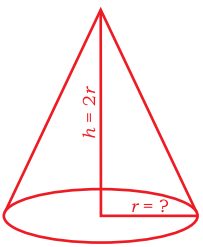
So, $A = \pi r^2$ or $154 = \frac{22}{7} r^2$

or $r^2 = 154 \times \frac{7}{22}$

or $r^2 = \frac{154 \times 7}{22}$ or $r^2 = \frac{\overset{7}{\cancel{154}} \times 7}{\cancel{22}_{11}} = 49$

or $r^2 = 7 \times 7$ or $r^2 = 7^2$

or $r = 7$cm



Thus, $r = 7$cm and $h = 3r = 3 \times 7 = 21$ cm

Volume of the cone: $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 7^2 \times 21 = \frac{1 \times 22 \times 7 \times 7 \times \cancel{21}^{\,1}}{\cancel{3}_{\,1} \times \cancel{7}^{\,1}}$$

$$= 22 \times 49 = 1078 \text{ cm}^3.$$

EXERCISE 9.6

1. Find the unknown elements of the cone in the table given below:

S. No.	Radius r	Height h	Slanting Height $s=\sqrt{h^2+r^2}$	Base Area of cone $=\pi r^2$	Curved Surface Area of Cone $=\pi rs$	Total Surface Area of Cone $=\pi r(r+s)$
(i)	21 cm	28 cm				
(ii)		56 cm	70 cm			
(iii)			32 cm			
(iv)		84 cm				
(v)	84 cm					33,264 cm^2

2. Find the unknown quantity in the table given below:

S. No.	Radius of Cone $= r$	Height of Cone $= h$	Slanting Height of Cone is: $s=\sqrt{h^2+r^2}$	Volume of Cone $= \frac{1}{3}\pi rh$
(i)	3 cm	$h = 7$ cm		
(ii)	3.5 cm	$h = 6$ cm		
(iii)		28 cm	35 cm	
(iv)		14 cm		154 cm^3
(v)	6.3 cm		105 cm	

3. The area of the base of a cone is 2464 cm^2. If its height is thrice the radius of the base, find its volume.

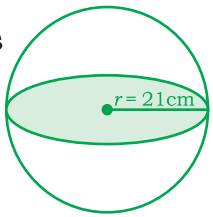
4. The volume of the cone is 352 cm^3. If its radius is 4cm, find the height of the cone.

5. If the radius of a cone is 3.5 metres and the height of the cone is 6 metres, find its volume.

9.3.3 Solve Real Life Problems Involving Surface Area and Volume of Sphere and Cone

Example 1. A solid copper sphere of radius 21 cm is melted and electric wire of diameter 0.42 cm is made out of the copper obtained. Find the length of the wire.

Solution: Given copper sphere of radius 21 cm is melted. So, the volume of the given copper sphere is:



Volume of sphere is: $V = \frac{4}{3}\pi r^3$, $r = 21\text{cm}$

$$= \frac{4}{\cancel{3}_1} \times \frac{22}{\cancel{7}_1} \times \cancel{21}^{\,1} \times 21 \times 21 = 88 \times 441 = 38{,}808\ cm^3$$

Now area of cross-section of wire $A = 4\pi\left(\frac{d}{2}\right)^2$

$$A = 4 \times \frac{22}{7} \times \left(\frac{0.42}{2}\right)^2 = \frac{\cancel{4}^{\,2} \times 22 \times 0.42 \times \cancel{0.42}^{\,0.06}}{\cancel{7}_1 \times \cancel{2}_1 \times \cancel{2}_1}$$

$$= 22 \times 0.42 \times 0.06 = 0.5544\ cm^2$$

$$\text{Length of wire} = \frac{\text{Volume of copper sphere}}{\text{Surface area of cross section of wire}}$$

$$= \frac{38808}{0.5544} = 70000\ \text{cm} = (\because 1\text{m} = 100\ \text{cm})$$

$$= \left(\frac{70000}{100}\right)\text{m} = 700\ \text{m}$$

Thus, the required length of the wire is 700 m.

Example 2. A tent in the form of a cone is 14 m high and its base is of radius 10.5 m. Find the area of canvas used to make the tent. Also find the volume of the tent.

Solution: Area of the curved surface of cone: $A = \pi rs = (\pi r\sqrt{h^2 + r^2})$

$$A = \left(\frac{22}{7} \times (14.0) \times \sqrt{(14)^2 + (10.5)^2}\right)$$

$$= \left(\frac{22}{7} \times 14.0 \times \sqrt{196 + 110.25}\right)$$

$$= \left(\frac{22}{7} \times 14.0 \times \sqrt{306.25}\right) m^2 = \left(\frac{22}{7} \times 14.0 \times 17.5\right)$$

$$= \left(\frac{22 \times \overset{2}{\cancel{14.0}} \times 17.5}{\underset{1}{\cancel{7}}}\right) m^2 = (44.0 \times 17.5) m^2 = 770\ m^2$$

Now volume of the cone, $V = \frac{1}{3}\pi r^2 h = \left(\frac{1}{3} \times \frac{22}{7} \times (10.5)^2 \times 14\right) m^3$

$$= \left(\frac{1 \times 22 \times \overset{0.5}{\cancel{\overset{3.5}{\cancel{10.5}}}} \times 10.5 \times 14}{\underset{1}{\cancel{3}} \times \underset{1}{\cancel{7}}}\right) m^3$$

$$= \left[(22 \times 0.5) \times (10.5 \times 14)\right] m^3 = (11 \times 147)\ m^3 = 1617\ m^3$$

Hence the area of the canvas required for the tent is 770 m^2

Also the volume of tent = 1617 m^3.

EXERCISE 9.7

1. What is the volume of a sphere if its radius 7 cm?

2 .A conical cup is full of ice cream. What will be the quantity of the ice cream; if the radius and height of the cone are 6cm and 10.5 cm, respectively?

3. A conical tent is 4.2 m high and its base is of radius 5.6 m. Find the volume of the air space in it.

4. How much sugar can a conical container hold whose height is 7 m and radius 6 m while 1 m^3 space contain 100 kg of sugar?

5. A conical shaped water tank has been planted in such a way that its circular region is in level with the earth's surface. The depth of tank is 7 m and radius of circular region is 3 m. How much litres of water will fill the tank?

6. A conical water tank is 6.3 m high and its base is of 8.4 m radius. What is the volume of the water tank ?

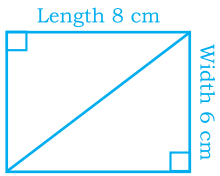
7. What is the radius of a spherical balloon, if its volume is $\frac{9\pi}{16}$ cm^3?

REVIEW EXERCISE 9

Select the correct answer.

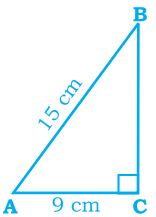
1. The length and width of a rectangle are 8 cm and 6c m respectively. The length of its diagonal is:

(i) 12 cm (ii) 16 cm (iii) 10 cm (iv) 14 cm



2. The given figure is a right angled triangle ABC. The length of the perpendicular is _____.

(i) 10 cm (ii) 12 cm (iii) 14 cm (iv) 16 cm



3. The statement of Hero's Formula is:

(i) $\triangle = \sqrt{S(S+a)(S+b)(S+c)}$ (ii) $\triangle = \sqrt{S(S-a)(S-b)(S-c)}$

(iii) $\triangle = \sqrt{S+(S-a)+(S-b)+(S-c)}$ (iv) $\triangle = \sqrt{S(S-a-b-c)}$

4. In Hero's Formula; the value of s is found as:

(i) $S = \frac{a-b-c}{2}$ (ii) $S = \frac{a+b-c}{2}$ (iii) $S = \frac{a+b+c}{2}$ (iv) $S = \frac{a+b+c}{3}$

5. The formula for finding the surface area of a sphere with radius r is:

(i) $A = \pi r^2$ (ii) $A = 3\pi r^2$ (iii) $A = 2\pi r^2$ (iv) $A = 4\pi r^2$

6. The formula for finding the volume of a sphere of radius r is:

(i) $V = \frac{2}{3}\pi r^2$ (ii) $V = \frac{4}{3}\pi r^2$ (iii) $V = \frac{3}{4}\pi r^3$ (iv) $V = \frac{4}{3}\pi r^3$

7. The value of π in a compound fraction is:

(i) $3\frac{1}{7}$ (ii) $3\frac{2}{7}$ (iii) $3\frac{3}{7}$ (iv) $3\frac{4}{7}$

8. In a cone $r = 1$m and $h = 3$ m, then its volume is:

(i) π m^3 (ii) π m^2 (iii) π m (iv) π

9. The radius of the base of a cone is 1m, then its base area is:

(i) π m (ii) π m^2 (iii) π m^3 (iv) π

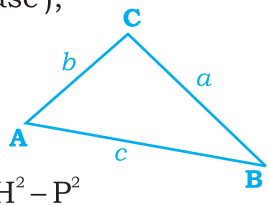
10. Let the slant height of a cone be 5 m and its radius be 2 m. Then the total surface area of the cone is:

(i) 11m^2 (ii) 22 m^2 (iii) 33 m^2 (iv) 44 m^2

SUMMARY

By Pythagoras theorem, in a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of other two sides (perpendicular and base),

I.e., $(\text{Hypotenuse})^2 = (\text{Perpendicular})^2 + (\text{Base})^2$



- The length of third side with the help of Pythagoras theorem can be found when the lengths of other two sides are given.

(i) $H^2 = P^2 + B^2$ (ii) $P^2 = H^2 - B^2$ (iii) $B^2 = H^2 - P^2$

- If the lengths of three sides of any triangle are a, b and c then the area of the triangle can be found by Hero's Formula:

$$\triangle ABC = \sqrt{S(S-a)(S-b)(S-c)} \quad \text{where } S = \frac{a+b+c}{2}$$

- The area of any quadrilateral can also be found by Hero's Formula.
- The surface areas of a sphere can be found by the formula

$S = 4\pi r^2$, where r is the radius of the sphere.

- The volume of a sphere is to be found by the formula

$V = \frac{4}{3}\pi r^3$ where r is the radius of sphere.

- A cone consists of two parts: (i) Circular Base (ii) Curved Surface. The Base Area of a cone is found by: πr^2

The curved surface of a cone is found by πrs, where r is radius and s is slant height.

The total surface area of a cone = Base Area + Curved Surface Area

$$= \pi r^2 + \pi rs = \pi r(r + s)$$

- The volume of the cone

$$= \frac{1}{3} \times (\text{Base area of cone}) \times (\text{Vertical height of cone})$$

Thus, $V = \frac{1}{3} \times \pi r^2 \times h$ or $V = \frac{1}{3}\pi r^2 h$

UNIT 10

DEMONSTRATIVE GEOMETRY

10.1 DEMONSTRATIVE GEOMETRY

(i) *Define Demonstrative Geometry*

'Demonstrative Geometry' is a branch of Mathematics in which statements concerning geometrical figures (called theorems) are proved through logical **Reasoning**.

The word 'Demonstrate' comes from the Latin word 'Demonstratum' which means **'to prove with certainty'**.

10.1.1 Reasoning

(ii) *Describe the Basics of reasoning*

Reasoning is the process of thinking about something in a logical way in order to form a conclusion or judgement. Reasoning may be **Inductive** or **Deductive**.

Quite often we come across situations where we draw conclusions based on observations, e.g., we visited a new country and noticed that each school bus we came across was of yellow colour. Upon our return we gave a concluding statement:

'All school busses in that country are of yellow colour.'

Such a reasoning, in which we **generalize** specific observations into universal statements, is called **Inductive Reasoning**.

The problem with inductive reasoning is that, even in the best (or strongest) cases the truth of the statement is not guaranteed; as in the above example there could have been some rare red buses which we couldn't have spotted in the whole trip.

Now consider the following statement:

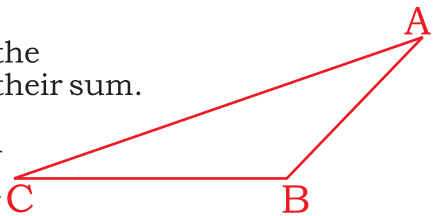
'Sum of the measures of three angles of a triangle is 180°.'

In order to prove this statement we can draw triangles of different kinds and different sizes. Measure their three angles. Then find their respective sum.

Activity: Let each student measure the three angles of this triangle and find their sum.

$m\angle A$ =_____, $m\angle B$ =_____

$m\angle C$ =_____, Sum = _____



We see that some students got the sum 179°. Others got the sum 181°. Only a few students got the sum 180°. Even one or two students got sum as 178° or 182°.

So from this activity we come to know that Inductive Reasoning is often not very reliable and we have to follow many precautionary measures to use this kind of reasoning. It is also time consuming.

Contrary to this, in **Deductive Reasoning** we deduce particular results from general ones. Generally we start from a set of statements already believed to be true and construct a series of deductions and use to establish proof for new statements.

We study Geometry through Deductive Reasoning because the new statements can be deduced with certainty.

The whole framework of deductive reasoning is based on the following four elements known as the *Basics of Reasoning.*

1. Undefined Terms: Some geometrical terms **are accepted without definition.** These are called **Undefined Terms.**

Point, Line, Plane and Space are ***Undefined Terms***.

2. Defined Terms: With the help of undefined terms, we define various new geometrical terms. For example: Line Segment, Ray, Angle, Triangle, Perpendicular, Bisector etc. are defined terms. Many of these terms have been defined in earlier classes.

3. Assumptions (or Fundamental Agreements):

Some mathematical statements are accepted without proof. These are called Assumptions or Fundamental Agreements. For example: *'Every number is equal to itself'* Or *'Two points determine a unique line.'* In section 10.1.2, we will introduce two types of assumptions and will enumerate all the assumptions that we shall use for reasoning in this unit.

4. Propositions: With the help of *Undefined Terms, Defined Terms,* and *Fundamental Agreements*, we can develop new statements. These statements are termed as propositions and may be proved true or false through logical reasoning. A generic proposition about some geometrical concept, if proved true, is called a theorem.

10.1.2 Axioms, Postulates and Theorem

(iii) *Describe the Types of Assumptions (Axioms and Postulates):*

Assumptions or Fundamental Agreements are of two kinds:

A. *Axioms:* Fundamental Agreements which are related to

numbers are called Axioms. Following are the axioms which will be used in this unit later on.

Axiom 1. Every number is equal to itself (Reflexive Property of Equations) i.e.,

$x = x \qquad \forall\, x \in \mathbb{R}$

or $\angle A \cong \angle A$ (Identity Congruence)

Axiom 2. Symmetric Property of Equations. i.e.,

$x = y \Leftrightarrow y = x \qquad \forall x, y \in \mathbb{R}$

Axiom 3. Transitive Property of Equations. i.e.,

$x = y$ and $y = z \Rightarrow x = z \qquad \forall x, y, z \in \mathbb{R}$

Axiom 4. Equals added to equals remain equal (Addition Property of Equations) i.e.,

$x = y \Rightarrow x + z = y + z \qquad \forall x, y, z \in \mathbb{R}$

Axiom 5. Equals subtracted from equals remain equal (Subtraction Property of Equations) i.e.,

$x = y \Rightarrow x - z = y - z \qquad \forall x, y, z \in \mathbb{R}$

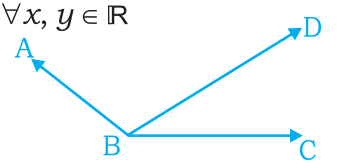
Axiom 6. If Equals are multiplied by equals, their products are equal (Multiplication Property of Equations) i.e.,

$x = y \Rightarrow xz = yz \qquad \forall x, y, z \in \mathbb{R}$

Axiom 7. If equals are divided by a non-zero number, their quotients are equal i.e.,

$x = y$ and $z \neq 0 \Rightarrow \frac{x}{z} = \frac{y}{z} \qquad \forall x, y \in \mathbb{R}$

Axiom 8. Whole is greater than its part. i.e.,



$m\angle ABC = m\angle ABD + m\angle CBD$

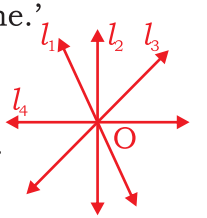
$\Rightarrow\ m\angle ABC > m\angle ABD,\ m\angle ABC > m\angle CBD$

B. *Postulates:* Fundamental Agreements which are related to geometrical figures are called Postulates.

Postulate 1. One and only one line passes through two distinct points. It is also stated as: 'Two points determine a line.'

Postulate 2. Countless lines can pass through a point.

Lines $l_1, l_2, l_3, \ldots, l_n$ are passing through a point O.



Postulate 3. One and only plane passes through three points not in a line.

Postulate 4. If two points of a line lie on a plane, the whole line lies on the plane.

[This postulate tells us that surface of plane is smooth and it is infinite on all sides.]

Postulate 5 (Distance Postulate). If A and B are any two points of a plane, then distance between them will be:

(i) 0 (zero), if A lies on B, i.e., A = B [read as A coincides with B]

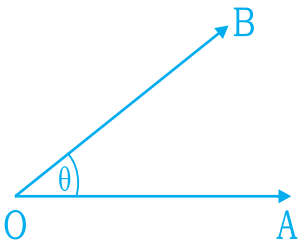
(ii) a 'Positive' real number when A and B are distinct points.

The distance from point A to point B is denoted as $|\overline{AB}|$ or $m\,\overline{AB}$, m stands for **'measure'** It may be noted that $m\,\overline{AB} = m\,\overline{BA}$ or $|\overline{AB}| = |\overline{BA}|$.

Postulate 6. A line segment can be bisected at one and only one point.

[This postulate tells that on a line segment AB, there is only one point (say P) between A and B such that $m\,\overline{AP} = m\,\overline{BP}$]

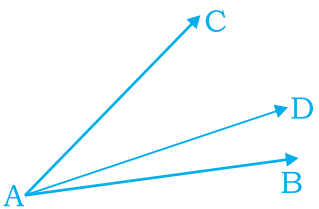
Postulate 7 (Angle Construction Postulate). If one arm of an angle is given, one and only one ray can be drawn on one side of it making an angle of given measure θ between 0° and 180°. i.e., $0^\circ < \theta < 180^\circ$.



Postulate 8 (Angle Addition Postulate). The sum of the measures of two adjacent angles is equal to the measure of the angle formed by their non-common arms.

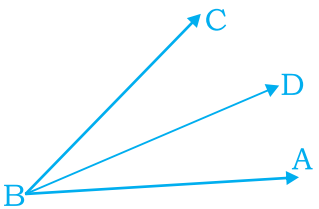
In this figure, ∠BAD and ∠CAD are two adjacent angles. So,

$m\angle BAD + m\angle CAD = m\angle BAC$ (or $m\angle CAB$).



Postulate 9 (Unique Bisector of an angle Postulate). One and only one bisector of an angle can be drawn. In this figure, $\overline{BD}$ is the bisector of ∠ABC.

So, $m\angle ABD = m\angle CBD = \frac{1}{2}\, m\angle ABC$



Postulate 10. At a given point in a line, one and only one perpendicular can be drawn to the line.

Postulate 11. Geometrical figures which can be made to coincide with one another are congruent.

Postulate 12. A geometrical figure may be moved from one position to another without any change in shape and size.

Postulate 13 (Playfair's Axiom). Through a point not on a line, one and only one line can be drawn parallel to the line. It is also stated as: 'Two intersecting lines cannot be parallel to a third line.'

(iv) *Describe parts of a proposition*

When a statement about any geometrical subject is proposed for discussion, it is called a ***proposition***. Propositions are either ***Theorems*** or ***Problems*** (also called ***Riders***).

A Proposition (Theorem or Rider) consists of two parts:

(i.) **Hypothesis** or that which is assumed to be true
(ii.) **Conclusion** or that which is to be proved.

(v) Describe the meanings of geometrical theorem, corollary and converse of a theorem.

(a) Theorem

A Theorem is a proposition in which some geometrical fact is stated. The general statement of a theorem is called Enunciation, e.g.,

'If two lines intersect, then the opposite vertical angles are congruent.'

In this proposition (i.e., theorem) ***Hypothesis*** is: "If two lines intersect" and ***Conclusion*** is: "then the opposite vertical angles are congruent." Usually, the ***hypothesis*** starts with 'If' and ***Conclusion*** with 'then'; but sometimes both ***hypothesis*** and ***Conclusion*** are mixed up in a single simple sentence, e.g.,

'Sum of the measures of the three angles of a triangle is 180°.'

Here the hypothesis is: 'If the angles are the three angles of a triangle' and **conclusion** is: 'then the sum of the measures of these angles is 180°.'

(b) Corollary

A Corollary (abbreviated as Cor.) is a fact which readily follows from a proved theorem. Such propositions are so trivial that we do not label them as separate theorems. These are the deductions made from a proposition or a theorem.

(c) Converse of a Theorem

One theorem is said to be the ***Converse*** of the other when the **hypothesis** of each is the **conclusion** of the other. In this text book, Theorems 1 and 2 are converse of one another. Also Theorems 7 and 8 are converse of one another.

In order **to prove a Theorem**, a Corollary or a Rider (given as an exercise) we follow certain fixed steps which are as under:

1. Figure or Diagram: According to the Enunciation, a neat figure is drawn showing the requirements of the Hypothesis as well as that of the Conclusion.

2. Given or Data: This is the particular enunciation of the hypothesis and is interpreted in terms of the figure in Step 1 above. It tells us what is the hypothesis in particular.

3. Required to Prove: Here the Conclusion is written in particular terms according to the figure drawn in Step 1 above.

4. Construction: Sometimes we need certain addition in the figure that helps us in proving the theorem. So we do certain construction.

5. Proof: This is the last part of a proof of a theorem (or a Problem) in which the given proposition is proved logically with the help of definitions, postulates, axioms, and the given data, or theorems proved earlier. We write statements one after other in one column and give reasons in another column opposite to it.

6. At the end of a Proof, we usually write **Q.E.D.**, the abbreviation of **Quod Erat Demonstrandum** meaning **'which was to be proved'**.

EXERCISE 10.1

1. Fill in the blanks with appropriate words:

(a) Demonstrative Geometry is a _____ of Mathematics in which _____ on geometry are proved through _____.

(b) Reasoning may be _____ or _____.

(c) In _____ reasoning, we draw conclusions based on observations.

(d) In _____ reasoning, we deduce particular results from general results and use _____ to prove new statements.

(e) Assumptions are those statements which are accepted _____.

(f) Fundamental Agreements are of two kinds. 1. _____ 2._____.

(g) Fundamental Agreements which are related to geometrical figures are called_____.

(h) The four basics of Reasoning are:

1._____ 2._____ 3._____ 4._____

2. Write 'True' or 'False' against each statement:

(a) One and only straight line can be drawn through three distinct points.

(b) If P and Q are coincident points, symbolically we write P = Q.

(c) Two points do not determine a line.

(d) Postulate 4 tells us that the surface of the plane is smooth and it is infinite on all sides.

(e) Complement of an angle of x° is $(90 - x)^\circ$.

(f) 'If two quantities are equal to the same quantity, they are equal to one another' is the same as Transitive Property of Equations.

3. Name the two parts of a Theorem or Proposition.

10.2 THEOREMS

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

Here proofs of prescribed theorems along with corollaries and their applications to solve appropriate problems are given:*

THEOREM 1

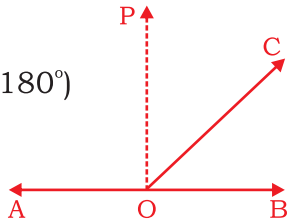
If a straight line stands on another straight line, the sum of measures of two angles so formed is equal to two right angles.

Given: $\overrightarrow{OC}$ stands on $\overleftrightarrow{AB}$

To prove : $m\angle AOC + m\angle BOC = 2rt\angle s$ (or 180°)

Construction: Through O, Draw $\overrightarrow{OP} \perp \overleftrightarrow{AB}$

i.e., $m\angle AOP = 1rt\angle = m\angle BOP$

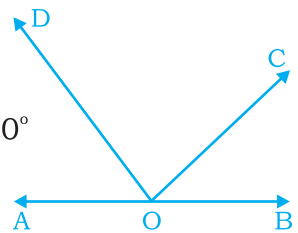


Proof: Statements	Reasons
1. $m\angle AOC = m\angle AOP + m\angle POC$	i. Angle Addition Postulate
2. $m\angle BOC = m\angle BOP - m\angle POC$	ii. $m\angle BOC + m\angle POC = m\angle POB$ (AAP)
3. $m\angle AOC + m\angle BOC = m\angle AOP + m\angle BOP$	iii. Adding (1) and (2), $m\angle POC$ cancels
$= 1rt\angle + 1rt\angle$	Each is a right angle
$= 2rt\angle s$ (or 180°)	
Q.E.D.	

Cor. 1. The sum of measures of all the angles at a point in a given straight line on the same side of it is equal to two right angles.

Given: $\overrightarrow{OC}$ and $\overrightarrow{OD}$ stand on $\overleftrightarrow{AB}$ on the same side of it making angles $\angle AOD, \angle DOC, \angle BOC$

To Prove : $m\angle AOD + m\angle DOC + m\angle BOC = 180^\circ$

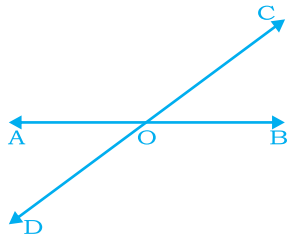


Note: Statements No. 1,2,3, etc have the reasons i, ii, iii, etc respectively, in each and every theorem.

Proof:

Statements	Reasons
1. $m\angle AOC + m\angle BOC = 180^\circ$	i. By Theorem 1
2. But $m\angle AOC = m\angle AOD + m\angle DOC$	ii. Angle Addition Postulate
3. $m\angle AOD + m\angle DOC + m\angle BOC = 180^\circ$	iii. Putting the value of $m\angle AOX$ in (1)
Q.E.D	

Cor. 2. If two lines intersect, the sum of the measures of the four angles is equal to four right angles.



Given: $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at O.

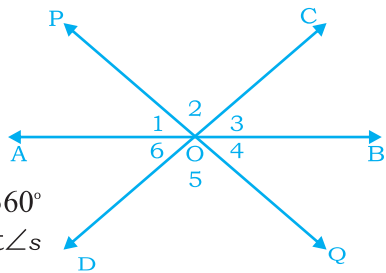
To Prove :

$m\angle AOC + m\angle BOC + m\angle BOD + m\angle AOD = 360^\circ$

Proof:

Statements	Reasons
1. $m\angle AOC + m\angle BOC = 180^\circ$	i. By Theorem 1
2. $m\angle BOD + m\angle AOD = 180^\circ$	ii. By Theorem 1
3. $m\angle AOC + m\angle BOC + m\angle BOD + m\angle AOD = 360^\circ$	iii. Adding (1) and (2)
Q.E.D.	

Cor. 3. If a number of straight lines meet at a point, the sum of measures of all the angles between successive lines is equal to four right angles.



Given: $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{PQ}$ intersect at pont O.

To Prove :

$m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ$

Proof: $= 4rt\angle s$

Statements	Reasons
1. $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$	i. By Theorem 1 Cor. 1
2. $m\angle 4 + m\angle 5 + m\angle 6 = 180^\circ$	ii. By Theorem 1 Cor. 1
3. $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 + m\angle 6 = 360^\circ = 4rt\angle s$	iii. Adding (1) and (2)
Q.E.D.	

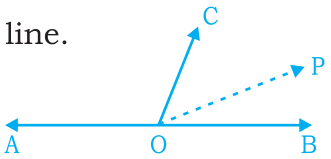
THEOREM 2

If the sum of measures of two adjacent angles is equal to two right angles, the external arms of the angles are in a straight line.

Given: Two adjacent angles $\angle AOC$ and $\angle BOC$ such that $m\angle AOC + m\angle BOC = 180^\circ = 2rt\angle s$

To prove : $\overrightarrow{OA}$, $\overrightarrow{OB}$ are in the same straight line.

Construction: If $\overrightarrow{OB}$ is not in the same straight line with $\overrightarrow{OA}$, produce $\overrightarrow{AO}$ to point P.



Proof:

Statements	Reasons
1. AOP is a straight line and OC stands on it	i. Construction, Supposition
2. $m\angle AOC + m\angle POC = 180^\circ$	ii. By Theorem 1
3. But $m\angle AOC + m\angle BOC = 180^\circ$	iii. Hypothesis
4. $m\angle AOC + m\angle POC = m\angle AOC + m\angle BOC$	iv. Axiom 3
5. Or $m\angle POC = m\angle BOC$	v. Canceling $m\angle AOC$ from both sides
6. Which is impossible unless OP and OB coincide Hence OA, OB are in the same straight line.	vi. Our Supposition in the construction is wrong. So, given facts are always true.

Q.E.D.

Note: Method used in the above proof is called **'Reduction ad Absurdum'** Method meaning 'Reduction to Absurdity' Method.

EXERCISE 10.2

1. In the figure of Theorem 1.

(i) Name the angle which is supplement of $\angle BOC$

(ii) Name the angle which is complement of $\angle BOC$

(iii) If $m\angle BOC = 50^\circ$, find measures of its complement and supplement.

2. In the figure of Theorem 1, Cor. 1, let $m\angle BOC = 45^\circ$, $m\angle AOD = 2x$, $m\angle COD = 3x$. Find the measures of $\angle AOC$, $\angle COD$.

[**Hint**: Solve for x the Eq: $2x + 3x + 45 = 180$] .

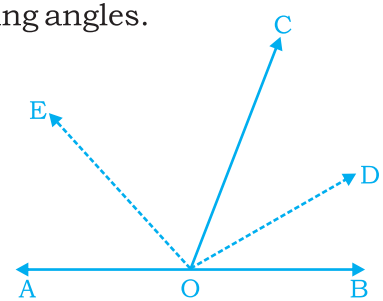
3. In the figure of Theorem 1 Cor. 2.

(a) Let $m\angle BOC = x = 40^\circ$, find the remaining angles

[Hint: Let $m\angle AOC = y$, $m\angle AOD = z$, $m\angle BOD = w$,
Now solve $w + 40^\circ = 180^\circ$, $y + 40^\circ = 180^\circ$, then $z + 140 = 180$ (why?)]

(b) Let $m\angle BOC = x = 45^\circ$, find the remaining angles.

4. Bisectors of two adjacent supplementary angles are perpendicular to each other.



Given: $\overrightarrow{OD}$, $\overrightarrow{OE}$ are bisectors of two adjacent angles AOC, BOC.

To Prove : $\overrightarrow{OD} \perp \overrightarrow{OE}$

Proof:

Statements	Reasons
1. $m\angle AOC + m\angle BOC = 180^\circ$	i. Given (Two adjacent suppl. $\angle s$)
2. $\frac{1}{2}\, m\angle AOC + \frac{1}{2}\, m\angle BOC = \frac{1}{2}(180^\circ)$	ii. Multiplying both sides of Eq. by $\frac{1}{2}$
3. or $m\angle COE + m\angle COD = 90^\circ$	iii. $\frac{1}{2}\, m\angle AOC = m\angle COE$ and $\frac{1}{2}\, m\angle BOC = m\angle COD$
4. or $m\angle EOD = 90^\circ$	iv. Angle Addition Postulate
5. or $OD \perp OE$	v. Two rays are perpendicular, If they form a $rt\angle$

Q.E.D.

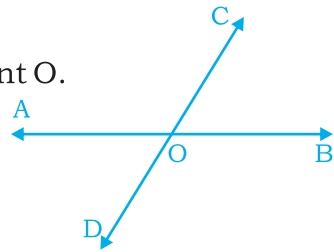
5. If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplementary angles. (Converse of problem 4 above).

[**Hint**: Since it is converse of problem 4, if we start from bottom of solution of problem 4 and end at the top with slight change somewhere for example you have to multiply by 2 at step 3.]

6. In the figure of problem 4, name all the pairs of complementary angles.

THEOREM 3

If two lines intersect each other, then the opposite vertical angles are congruent.



Given: Two lines AB and CD intersect at point O.

To Prove:

(i) $\angle BOC \cong \angle AOD$ (ii) $\angle AOC \cong \angle BOD$

Proof:

Statements	Reasons
1. $m\angle AOC + m\angle BOC = 180^\circ$	i. By Theorem 1
2. $m\angle AOD + m\angle AOC = 180^\circ$	ii. By Theorem 1
3. $\therefore m\angle AOC + m\angle BOC = m\angle AOD + m\angle AOC$	iii. Transitive Prop. of Eq.
4. $\therefore m\angle BOC = m\angle AOD$	iv. Cancelling $m\angle AOC$ from both sides
5. or $\angle BOC \cong \angle AOD$	v. If two angles are equal in measure, they are congruent.
6. Similarly, we can prove that $\angle AOC \cong \angle BOD$	vi. By the above process.
Q.E.D.	

EXERCISE 10.3

1. In the figure of Theorem 3, if $m\angle BOC = 70^\circ$, find the measures of other angles.

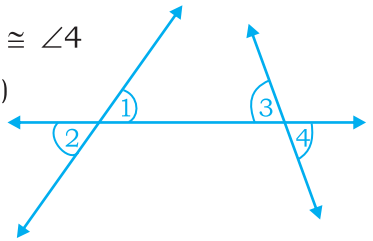
[**Comment:** In the previous exercise we used Theorem 1 to find other angles, because until then we had not learnt Theorem 3. But now we can use Theorem 3 together with Theorem 1]

Solution: $m\angle AOD = m\angle BOC = 70^\circ$ (By Theorem 3) and $\angle AOC$ is the supplement of $\angle BOC$. Therefore, $m\angle AOC = 180^\circ - 70^\circ = 110^\circ$ (By Theorem 1) and $m\angle BOD = 110^\circ$ (By Theorem 3).

2. Draw two lines intersecting one another at an angle of 30°, find the measures remaining angles as discussed above.

3. In this figure $\angle 1 \cong \angle 3$, Prove that $\angle 2 \cong \angle 4$

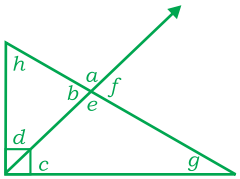
[**Hint:** First prove $\angle 2 \cong \angle 1$ and $\angle 3 \cong \angle 4$ (Th. 1) then use Axioms3 since $\angle 1 \cong \angle 3$ is given.]



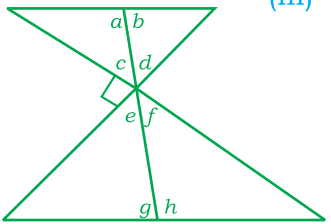
4. If the bisectors of a pair of congruent angles having the same vertex are two opposite rays, then the sides of the angles are two intersecting lines.

5. In the following figures, name the complementary angles, the supplementary angles, and the vertically opposite angles:

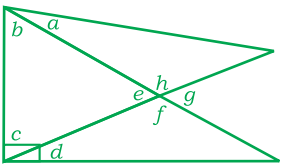
(i)



(ii)

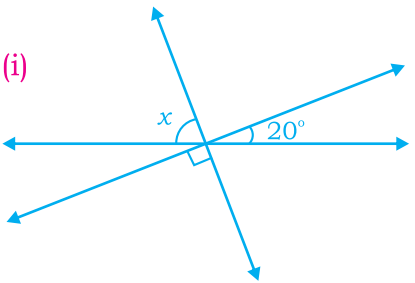


(iii)

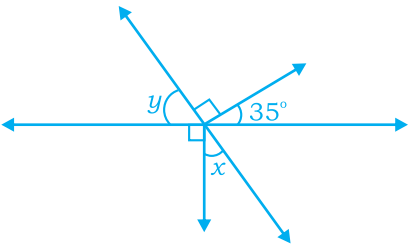


6. Find the degree measure for the angles marked x, y and z.

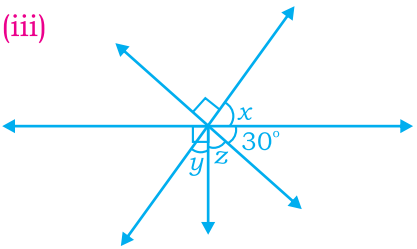
(i)



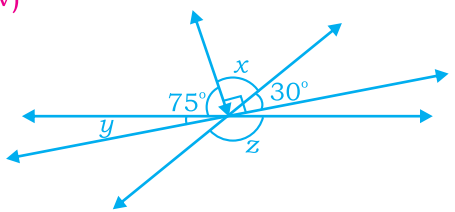
(ii)



(iii)



(iv)



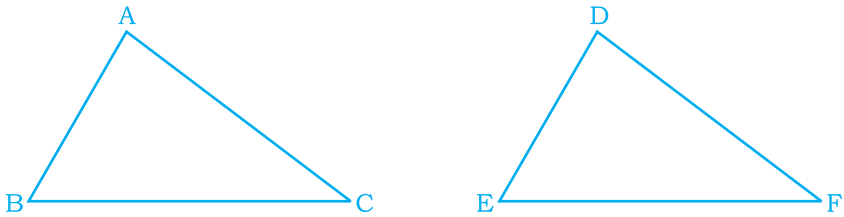
Definitions:

One-One Correspondence between two triangles.

Every triangle has three vertices and three sides, so it is always possible to establish One-One Correspondence between the sides, angles and vertices of two triangles.

Symbol '↔' is used for One-One (or 1 - 1) Correspondence.

The statement 'ΔABC ↔ ΔDEF' means that: A corresponds to D, B to E, C to F, $\overline{AB}$ corresponds to $\overline{DE}$, $\overline{BC}$ to $\overline{EF}$, and $\overline{AC}$ to $\overline{DF}$.



'ΔABC ↔ ΔEDF' is not the same as 'ΔABC ↔ ΔDEF' because here A corresponds to E, B to D, and so on.

Congruence of Triangles:

Two Triangles are said to be congruent if their corresponding angles and sides are congruent. Conversely, if two triangles are congruent, then their corresponding angles and sides are congruent. Symbol used for congruence is '≅'.

Note 1. If one correspondence of two triangles is a congruence, it is not necessary that other correspondences between them are also congruences.

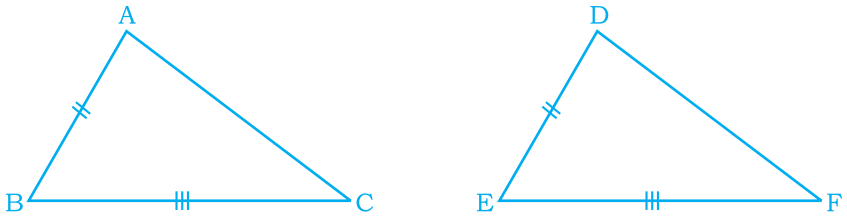
Note 2. A triangle is congruent to itself. This is called Identity Congruence of triangles.
For example: ΔABC ≅ ΔABC

Note 3. ΔABC ≅ ΔDEF implies ΔDEF ≅ ΔABC (Symmetric Property of Congruence).

Note 4. If ΔABC ≅ ΔDEF and ΔDEF ≅ ΔPQR, then ΔABC ≅ ΔPQR (Transitive Property of Congruence) .

THEOREM 4

In any correspondence of two triangles, if two sides and included angle of one triangle are congruent to the corresponding sides and included angles of the other, the two triangles are congruent.



Given: In $\Delta ABC \leftrightarrow \Delta DEF$

(i) $\overline{AB} \cong \overline{DE}$ **(ii)** $\angle B \cong \angle E$ **(iii)** $\overline{BC} \cong \overline{EF}$

To Prove : $\Delta ABC \cong \Delta DEF$

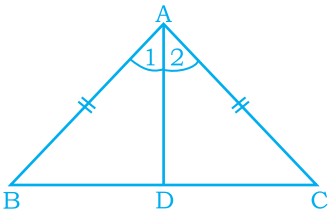
Proof:

Statements	Reasons
1. Apply ΔABC on ΔDEF so that Point B falls on point E and BC along EF	i. Postulate 12
2. $BC \cong EF$	ii. Given
3. $\therefore$ Point C coincides with point F	iii. By definition of Congruence
4. $\angle B \cong \angle E$	iv. Given
5. $\therefore$ BA falls along ED	v. By Angle Construction Postulate
6. $AB \cong DE$	vi. Given
7. $\therefore$ A must coincide with point D	vii. By definition of Congruence
8. ΔABC coincides with ΔDEF	viii. As proved above
9. Hence, $\Delta ABC \cong \Delta DEF$.	ix. By Postulate 11
Q.E.D	

Note: This theorem is also quoted as 'S.A.S. $\cong$ S.A.S.'

THEOREM 5

If two sides of a triangle are congruent, then the angles opposite to these sides are congruent.



Given: In ΔABC, $\overline{AB} \cong \overline{AC}$

To Prove : $\angle B \cong \angle C$

Construction: Draw $\overline{AD}$ the bisector of $\angle A$ meeting $\overline{BC}$ at point D.

Proof:

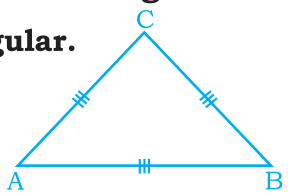
Statements	Reasons
1. In $\Delta ABD \leftrightarrow \Delta ACD$	i. 1-1 correspondence between two Δs
(A) $AB \cong AC$	(a) Given
(B) $\angle 1 \cong \angle 2$	(b) Construction
(C) $AD \cong AD$	(c) Common (Identity Congruence)
2. $\Delta ABD \cong \Delta ACD$	ii. S.A.S. $\cong$ S.A.S.
3. $\therefore \angle B \cong \angle C$	iii. By the congruence of triangle

Q.E.D.

Note: Theorem 5 can also be stated as:

'In an Isosceles triangle, angles at the base are congruent.'

Cor. 1. An equilateral triangle is also equiangular.



Given: ΔABC in which $\overline{AB} \cong \overline{BC} \cong \overline{AC}$

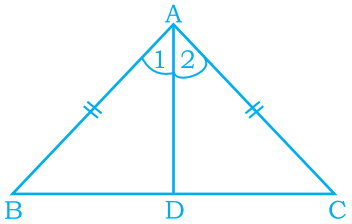
To Prove : $\angle A \cong \angle B \cong \angle C$

Proof:

Statements	Reasons
1. In ΔABC, $BC \cong AC$	i. Given
2. $\therefore \angle A \cong \angle B$	ii. By Theorem 5
3. Similarly, $\angle B \cong \angle C$	iii. $AB \cong AC$
4. $\therefore \angle A \cong \angle B \cong \angle C$	iv. Combining (2) and (3)

Q.E.D.

Cor. 2. In an Isosceles triangle, the bisector of the angle at the vertex is the right bisector of the base.



Given: In ΔABC, $\overline{AB} \cong \overline{AC}$, and $\overline{AD}$ is the bisector of $\angle A$ meeting $\overline{BC}$ at D.

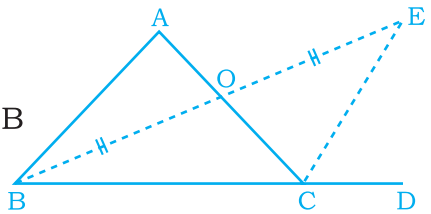
To Prove : $\overline{BD} \cong \overline{DC}$, and $\overline{AD} \perp \overline{BC}$

Proof:

Statements	Reasons
1. In $\Delta ABD \leftrightarrow \Delta ACD$	I. 1-1 correspondence between two Δs.
(A) $AB \cong AC$	(a) Given
(B) $\angle 1 \cong \angle 2$	(b) Given
(C) $AD \cong AD$	(c) Common
2. $\therefore$ $\Delta ABD \cong \Delta ACD$	ii. By Theorem 4 (S.A.S. $\cong$ S.A.S.)
3. $\therefore$ $BD \cong DC$ and $\angle ADB \cong \angle ADC$	iii. By the congruence of triangles
4. But $m\angle ADB + m\angle ADC = 180^\circ$	iv. By Theorem 1
5. $\therefore$ $m\angle ADB = 90^\circ = m\angle ADC$	v. By Definition of rt. angle
6. $\therefore$ $AD \perp BC$	vi. By Definition of Perpendicular
7. $\therefore$ AD is Perp. Bisector of BC	vii. AD bisects BC at right angles
Q.E.D.	

THEOREM 6

An exterior angle of a triangle is greater in measure than either of its opposite interior angles.



Given: ΔABC with exterior angle ACD

To Prove :

$m\angle ACD > m\angle A$, $m\angle ACD > m\angle B$

Construction:

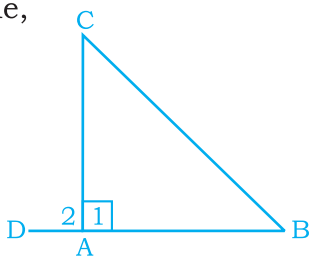
Let point O be the mid-point of $\overline{AC}$.

Draw $\overline{BO}$ and produce it to point E. So that, $m\,\overline{BO} = m\,\overline{OE}$. Join point C to point E.

Proof:

Statements	Reasons
1. In $\Delta AOB \leftrightarrow \Delta COE$	I. 1-1 correspondence between two Δs.
(A) $BO \cong OE$	(a) Construction
(B) $\angle AOB \cong \angle COE$	(b) Vertical angles
(C) $AO \cong CO$	(c) Construction
2. $\therefore$ $\Delta AOB \cong \Delta COE$	ii. S.A.S. $\cong$ S.A.S.
3. $\therefore$ $m\angle A = m\angle ACE$	iii. By the congruence of triangles
4. But $m\angle ACD = m\angle ACE + m\angle ECD$	iv. Angle addition Postulate
5. $\therefore$ $m\angle ACD > m\angle ACE$	v. Whole is greater than its part
6. $\therefore$ $m\angle ACD > m\angle A$	vi. $m\angle A = m\angle ACE$ (Proved above)
7. Similarly, $m\angle ACD > m\angle B$	
Q.E.D.	

Cor. 1. In a triangle, if one angle is a right angle, then the other angles are acute angles.



Given: ΔABC is a right-triangle at Point A.

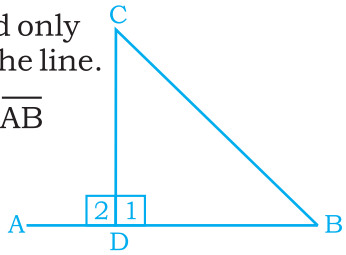
To Prove : $m\angle B < 90^\circ$ and $m\angle C < 90^\circ$

Construction: Produce BA to any point D.

Proof:

Statements	Reasons
1. $m\angle 2 = 90^\circ$	i. $m\angle 1 + m\angle 2 = 180^\circ$ and $m\angle 1 = 90^\circ$
2. But $\angle 2$ is exterior angle of ΔABC	ii. By Definition
3. $\therefore$ $m\angle B < m\angle 2 \Rightarrow m\angle B < 90^\circ$	iii. By Theorem 6
4. Similarly, $m\angle C < 90^\circ$	iv. By the above reason
Q.E.D.	

Cor. 2. From a point not on the line, one and only one perpendicular can be drawn to the line.



Given: C is any point outside $\overline{AB}$, and $\overline{CD} \perp \overline{AB}$

To Prove : CD is the only perpendicular.

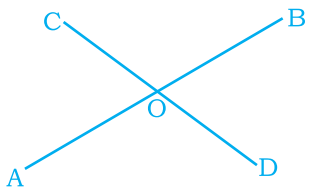
Proof:

Statements	Reasons
1. If CD is not the only perpendicular. Suppose CB is also perpendicular on AB	i. Supposition
2. But in ΔDBC, $m\angle B < \angle 2$	ii. By Theorem 6
3. $\Rightarrow$ $m\angle B < 90^\circ$	iii. $m\angle 2 = 90^\circ$
4. $\therefore$ CB is not perpendicular on AB	iv. Our supposition is wrong
5. Hence CD is the only perpendicular on AB	v. As proved above .
Q.E.D	

EXERCISE 10.4

1. In the adjacent figure, O is the mid point of line segments AB and CD.

Prove that $\overline{AC} \cong \overline{BD}$



[**Hint:** Join point A to point C and point B to point D. Prove $\Delta AOC \cong \Delta BOD$ by S.A.S. $\cong$ S.A.S.]

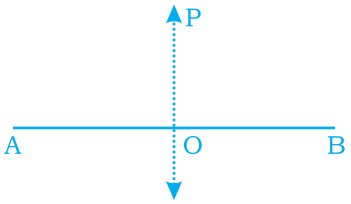
2. Two line segments bisect one another. Prove that the line segments joining their end-points are congruent. [This is the general statement of problem 1 above]

3. Prove that the diagonals of a rectangle are congruent.

[**Hint:** Let $\overline{AC}$ and $\overline{BD}$ be the diagonals of rectangle ABCD. Prove that $\Delta ABC \cong \Delta BAD$ by S.A.S. Theorem]

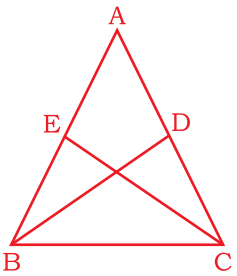
4. In an Isosceles triangle, the bisector of the angle at the vertex angle is the right bisector of the base. Prove.

5. Every point on the perpendicular bisector of a line segment is equidistant from the end-points of the line segment.



[Here $\overleftrightarrow{OP}$ is the perpendicular bisector of $\overline{AB}$. We have to prove that $\overline{AP} \cong \overline{BP}$. So join $\overline{AP}$ and $\overline{BP}$, then prove that triangles AOP and BOP are congruent by S.A.S. Theorem]

6. Medians to the congruent sides of an isosceles triangle are congruent.



Given: In ΔABC, $\overline{AB} \cong \overline{AC}$ and $\overline{BD}$, $\overline{CE}$ are medians

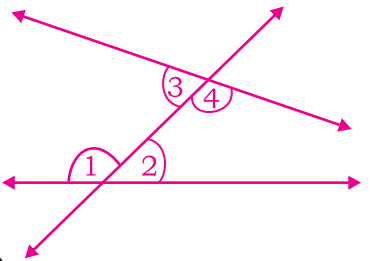
To Prove That: $\overline{BD} \cong \overline{CE}$

Proof:

Statements	Reasons
1. In $\Delta ABD \leftrightarrow \Delta ACE$	i.
(i) $AB \cong AC$	(a) Given
(ii) $\angle A \cong \angle A$	(b) Common
(iii) $AD \cong AE$	(c) Halves of congruent sides are $\cong$
2. $\therefore$ $\Delta ABD \cong \Delta ACE$	ii. S.A.S. $\cong$ S.A.S.
3. $\therefore$ $BD \cong CE$	iii. By the congruence of triangles
Q.E.D.	

7. From a point not on the line, one and only one perpendicular can be drawn to the line.

8. Consider the figure given here and discuss the following questions/statemens with your class-mates and try to find out the answers of the questions asked here.



(i). What relationship do you suggest between the pair of angles 1, 2 and 3, 4?

(ii). Are $\angle 1$ and $\angle 4$ congruent? Are $\angle 3$ and $\angle 2$ congruent? If not, why? Under what Conditions they may be congruent?

(iii). If $\angle 3$ and $\angle 2$ are congruent, what will happen to the two converging lines?

(iv). $\angle 1, \angle 2$ and $\angle 3, \angle 4$ are supplementary angles. Do you agree? Give reasons. Are $\angle 1, \angle 3$ and $\angle 2, \angle 4$ also supplementary angles. If not, suggest a condition when these may be supplementary.

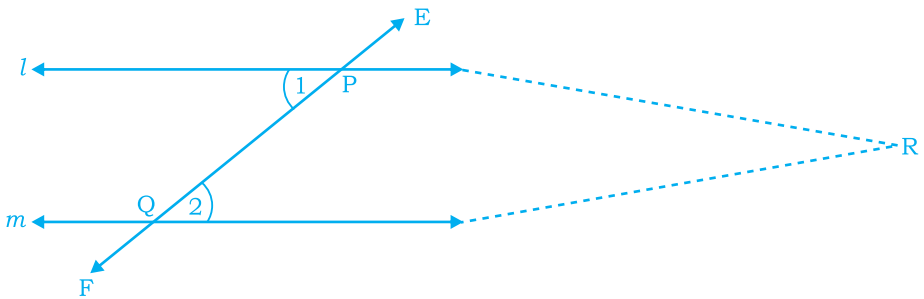
(v). Find the condition when the pair of interior angles on the same side of a transversal are supplementary.

(vi). If the converging lines are produced, they will meet at some point. What new figure will be developed? Out of four interior angles of converging lines which two of them will become the exterior angles of the triangle? Which exterior angle is greater than $m\angle 2$, and which exterior angle is greater than $m\angle 4$?

We will get answers to some of these questions after studying Theorems 7 and 8, so revisit these questions after we have learnt these theorems.

THEOREM 7

If a transversal intersects two coplanar lines such that the pair of alternate angles are congruent, then the lines are parallel.



Given: l and m are two coplanar lines and transversal $\overleftrightarrow{EF}$ intersects them at P and Q respectively, such that $\angle 1 \cong \angle 2$ (Pair of Alternate angles)

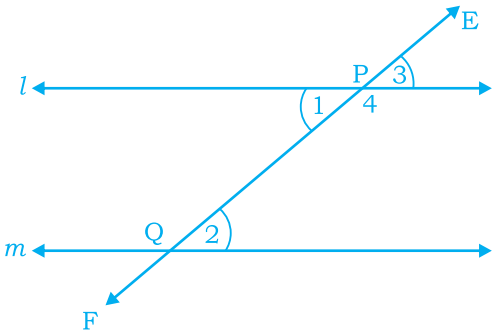
To Prove : $l \| m$

Proof: If l and m are parallel, then being coplanar they will meet at some point (say R) and PQR will be a triangle.

Statements	Reasons
1. In ΔPQR, $\angle 1$ is exterior angle, and $\angle 2$ is interior opposite angle	i. By definition
2. $\therefore$ $m\angle 1 > m\angle 2$	ii By Theorem 6
3. But $m\angle 1 = m\angle 2$	iii. Given
4. Statements 2 and 3 cannot be true simultaneously	iv. Trichotomy Prop. of Order Rel.
5. So $m\angle 1 = m\angle 2$ is true and l, m do not meet.	v. Our supposition is wrong.
6. or $l \parallel m$	vi. They do not intersect
Q.E.D.	

Cor. 1. If a transversal intersects two coplanar lines such that the pair of corresponding angles are congruent then the lines are parallel.

Given: l and m are two coplanar lines and transversal $\overleftrightarrow{EF}$ intersects them at points P and Q respectively, such that $\angle 3 \cong \angle 2$ (Pair of Corresponding angles).



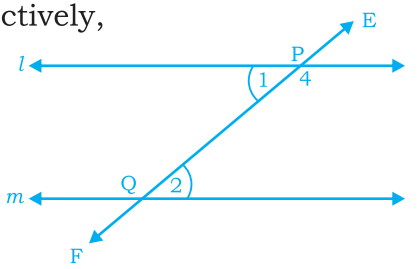
To Prove $l \parallel m$

Proof:

Statements	Reasons
1. $m\angle 1 = m\angle 3$	i. By Theorem 3
2. But $m\angle 3 = m\angle 2$	ii. Given
3. $\therefore$ $m\angle 1 = m\angle 2$	iii. Transitive Prop. of Eq.
4. But these are alternate $\angle$s, $\therefore$ $l \parallel m$	iv. By Theorem 7
Q.E.D.	

Cor. 2. If a transversal intersects two coplanar lines such that the interior angles on the same side of the transversal are supplementary, then the lines are parallel.

Given: l and m are two coplanar lines and transversal $\overleftrightarrow{EF}$ intersects them at points P and Q respectively, such that $m\angle 2 + m\angle 4 = 180^\circ$ (Pair of supplementary angles)



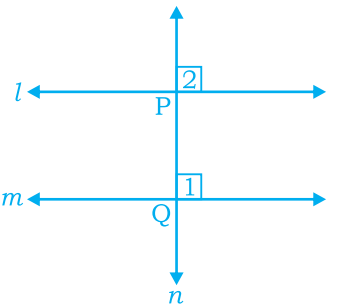
To Prove : $l \parallel m$

Proof:

Statements	Reasons
1. $\angle 1$ is the supplement of $\angle 4$	i. By Theorem 1
2. But $\angle 2$ is the supplement of $\angle 4$	ii. Given
3. $\therefore$ $m\angle 1 = m\angle 2$	ii. Supplements of same angle are $\cong$
4. But these are alternate $\angle$s, $\therefore$ $l \parallel m$	iv. By Theorem 7
Q.E.D.	

Cor. 3. In a plane, if a line is perpendicular to each of two lines, then the two lines are parallel.

Give: l and m are two coplanar lines and transversal n intersects them at points P and Q respectively, such that $m\angle 1 = 90^\circ = m\angle 2$



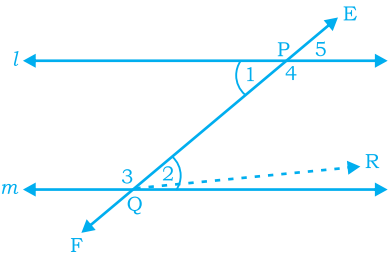
To Prove: $l \parallel m$

Proof:

Statements	Reasons
1. $m\angle 1 = m\angle 2$	i. Each is a right angle (Given)
2. But these are Corresponding $\angle$s, $\therefore$ $l \parallel m$	ii. By Theorem 7 Corresponding $\angle$s
Q.E.D.	

THEOREM 8

If a transversal intersects two parallel lines, the alternate angles so formed are congruent.



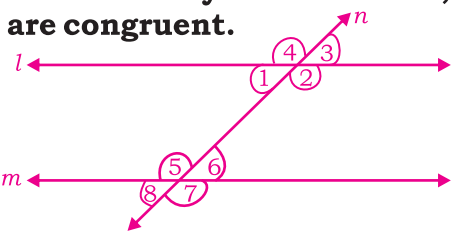
Given: l and m are two parallel lines and transversal $\overleftrightarrow{EF}$ intersects them at points P and Q respectively.

To Prove : (i) $\angle 1 \cong \angle 2$ (ii) $\angle 3 \cong \angle 4$

Proof: Suppose $\angle 1$ is not congruent to $\angle 2$, but $\angle 1 \cong \angle PQR$, where $\overrightarrow{QR}$ does not lie on m.

Statements	Reasons
1. Since $\angle 1 \cong \angle PQR$	i. By supposition
2. $\therefore$ $l \parallel QR$	ii. By Theorem 7
3. But $l \parallel m$	iii. Given
4. $\therefore$ l is parallel to two lines m and QR which is impossible	iv. By Playfair's Axiom
5. $\therefore$ $\angle 1 \cong \angle 2$	v. Our supposition leads to absurd result
6. Similarly, $\angle 3 \cong \angle 4$	vi. By the above process
Q.E.D	

Cor. 1. If two parallel lines are intersected by a transversal, each pair of corresponding angles are congruent.



Given: $l \parallel m$ and transversal n intersects them forming eight angles as shown in the figure.

To Prove : $\angle 1 \cong \angle 8, \angle 2 \cong \angle 7, \angle 3 \cong \angle 6, \angle 4 \cong \angle 5$

(Pairs of Corresponding angles)

Proof:

Statements	Reasons
1. $l \parallel m$	i. Given
2. $\therefore$ $\angle 1 \cong \angle 6$	ii. Alt. $\angle$s (Theorem 8)
3. But $\angle 1 \cong \angle 3$	iii. Vertical Angles
4. $\therefore$ $\angle 6 \cong \angle 3$	iv. Transitive Prop. of Eq.
5. Similarly, $\angle 1 \cong \angle 8$, $\angle 7 \cong \angle 2$, $\angle 5 \cong \angle 4$	v. By the above process
Q.E.D.	

Cor. 2. If two parallel lines are intersected by a transversal, the interior angles on the same side of the transversal are supplementary.

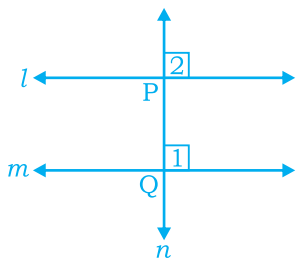
Given: $l \parallel m$ and transversal n intersects them forming eight angles as shown in the figure. (See the figure of Cor. 1)

To Prove : $m\angle 1 + m\angle 5 = 180^\circ$, $m\angle 2 + m\angle 6 = 180^\circ$

Proof:

Statements	Reasons
1. $l \parallel m$	i. Given
2. $\therefore$ $m\angle 6 = m\angle 1$	ii. Alt. $\angle$s (Theorem 8)
3. or $m\angle 6 + m\angle 2 = m\angle 1 + m\angle 2$	iii. Adding $m\angle 2$ on both sides
4. $\therefore$ $m\angle 6 + m\angle 2 = 180^\circ$	iv. $m\angle 1 + m\angle 2 = 180^\circ$ (Th. 1)
5. Simimilarly $m\angle 1 + m\angle 5 = 180^\circ$	v. By the above process
Q.E.D.	

Cor. 3. In a plane, if a line is perpendicular to one of the two parallel lines, it is perpendicular to the other line also.



Given: $l \parallel m$ and n is perpendicular on m

To Prove : n is also perpendicular on l

Proof:

Statements	Reasons
1. $l \parallel m$	i. Given
2. $\therefore$ $\angle 2 \cong \angle 1$ implies $\angle 2$ is a rt. $\angle$	ii. By Th. 8, Cor. 1 (Corresponding angles)
3. $\therefore$ n is also perpendicular on l	iii. $\angle 2$ is a rt. angle as proved

Q.E.D.

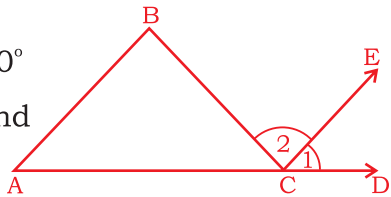
THEOREM 9

The sum of measures of the three angles of a triangle is 180°.

Given: ΔABC

To Prove : $m\angle A + m\angle B + m\angle C = 180^\circ$

Construction: Produce $\overrightarrow{AC}$ to point D and Draw $\overrightarrow{CE}$ parallel to $\overline{AB}$.



Proof:

Statements	Reasons
1. $\overline{AB}$ is parallel to CE, $\overrightarrow{AD}$ is transversal	i. Construction
2. $m\angle A = m\angle 1$	ii. Corresponding angles
3. $\overline{AB}$ is parallel to $\overrightarrow{CE}$, $\overline{BC}$ is transversal	iii. Construction
4. $m\angle B = m\angle 2$	iv. Alternate angles
5. $m\angle A + m\angle B = m\angle 1 + m\angle 2 = m\angle BCD$	v. Adding (2) and (4), Angle Addition Postulate
6. Adding $m\angle ACB$ to both sides $m\angle A + m\angle B + \angle ACB = m\angle BCD + m\angle ACB$	vi. Addition Prop. of Eqs.
7. or $m\angle A + m\angle B + m\angle C = 180^\circ$	vii. $m\angle BCD + m\angle ACB = 180^\circ$

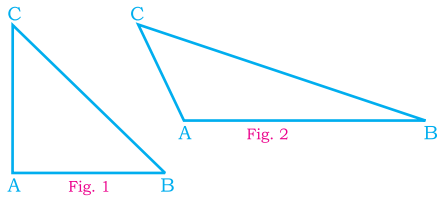
Q.E.D

Cor. 1. In a triangle there can be only one right angle or one obtuse angle.

Given: ΔABC, $m\angle A = 90^\circ$

Or $m\angle A > 90^\circ$

To Prove : ΔABC has only one Right angle or only One Obtuse angle.



Proof:

Statements	Reasons
1. In Fig. 1, $m\angle A = 90^\circ$	i. Given
2. $\therefore$ $m\angle B + m\angle C = 180^\circ - 90^\circ = 90^\circ$ i.e., $\angle B$ and $\angle C$ are complementary $\angle$s	ii. By Theorem 9 By Def. of Compl. Angles
3. $\therefore$ Each of $\angle B$ and $\angle C$ is less than a rt. $\angle$ or ΔABC has only one right angle	iii. Their sum is 90° iv. Given
4. In Fig. 2, $m\angle A > 90^\circ$	v. Sum of measures of three angles is 180°
5. $\therefore$ $m\angle B + m\angle C < 90^\circ$	
6. $\therefore$ Each of $\angle B$ and $\angle C$ is less than a rt. $\angle$ or ΔABC has only one obtuse angle.	vi. Sum of measures of these two angles is less than 90°

Q.E.D.

Cor. 2. Every triangle has at least two acute angles.

Given: ΔABC (Refer the figures given above)

To Prove That: ΔABC has at least two acute angles

Proof:

Statements	Reasons
1. If ΔABC has only one Right or only one Obtuse angle, then it has two acute angles	i. By Theorem 9 Cor. 1
2. If ΔABC has no Right or no Obtuse angle, then it has all the three angles acute	ii. A triangle has 3 angles
3. Hence, every triangle has at least two acute $\angle$s.	iii. As proved above

Q.E.D.

Cor. 3. In a right-angled triangle, acute angles are complementary. [For its solution, follow the first two steps of the proof of Cor. 1]

Cor. 4. From a point not on the line, one and only one perpendicular can be drawn to a given line. [Proved in Th. 6, Cor. 2]

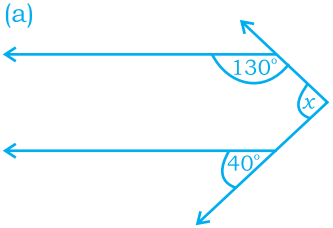
Cor. 5. The measure of an exterior angle of a triangle is equal to the sum of the measures of non-adjacent interior angles of the triangle. [For solution, draw the figure of Th. 9, write 'Given' etc yourself, then

Cor. 6. If two angles of a triangle are congruent to two angles of another triangle, then it third angle will be congruent to the third angle of the other triangle [Left as an exercise for the students]

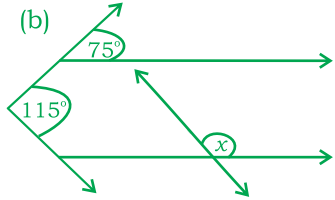
EXERCISE 10.5

1. In each of the figures given below, find the angle marked x. Numbers indicate the degree measure of the angle.

(a)

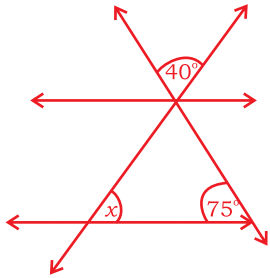


(b)



(c)

(d)

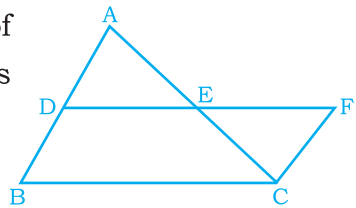


[**Hint**: From (a) to (d) through the vertex of angle marked x (or 135°) draw a line parallel to both the lines]

2. If two line segments AB and CD bisect one another at point O, prove that $\overline{AC}$ is parallel to $\overline{BD}$ and $\overline{AD}$ is parallel to $\overline{BC}$. [This problem has previously been solved. After proving the two triangles congruent use Theorem 7 to prove the lines to be parallel using alternate angles congruent.]

3. Points D and E are the mid-points of the sides $\overline{AB}$ and $\overline{AC}$ of a ΔABC. If $\overline{DE}$ is produced to point F so that $\overline{EF} \cong \overline{DE}$, prove that $\overline{CF} \parallel \overline{AB}$ and $\overline{DE} \parallel \overline{BC}$.

Given: Points D and E are the mid-points of the sides $\overline{AB}$ and $\overline{AC}$ of ΔABC. $\overline{DE}$ is produced to point F so that $\overline{EF} \cong \overline{DE}$



To Prove That:
$\overline{CF} \parallel \overline{AB}$ and $\overline{DE} \parallel \overline{BC}$.

Proof:

Statements	Reasons
1. In $\Delta ADE \leftrightarrow \Delta CFE$	i. 1-1 correspondence between two Δs.
(A) $\overline{AE} \cong \overline{CE}$	(a) Given (E is the mid-point)
(B) $\angle AED \cong \angle CEF$	(b) Vertical angles
(C) $\overline{DE} \cong \overline{EF}$	(c) Given
2. $\therefore$ $\Delta ADE \cong \Delta CFE$	ii. S.A.S. $\cong$ S.A.S.
3. $\therefore$ $\angle A \cong \angle ECF$ and $\overline{CF} \cong \overline{AD} \cong \overline{BD}$	iii. By the congruence of triangles
4. But $\angle A$ and $\angle ECF$ are alt. $\angle$s $\therefore$ $\overline{CF} \parallel \overline{AB}$ or $\overline{CF} \parallel \overline{BD}$	iv. By Theorem 7
5. $\overline{CF} \cong \overline{BD}$ and $\overline{CF} \parallel \overline{BD}$	v. Proved above
6. $\therefore$ BCFD is a parallelogram	vi. One pair of opposite sides of a quadrilateral is parallel and congruent, it is a parallelogram
7. $\therefore$ $\overline{DF} \parallel \overline{BC}$ or $\overline{DE} \parallel \overline{BC}$	vii. Opposite sides of parallelogram are parallel
8. or $\overline{CF} \parallel \overline{AB}$ and $\overline{DE} \parallel \overline{BC}$	viii. As proved above
Q.E.D	

4. Prove that opposite angles of a prallelogram are congruent.
[**Hint**: Use Cor. 2 of Theorem 8, i.e., $m\angle A + m\angle B = 180^\circ$ etc.]

5. How many degrees are there in an angle of an equilateral triangle?

6. Find measure of an angle which is equal to the sum of measures of other two angles of a triangle.
[**Hint**: By Theorem 9, Cor. 3, the sum of other two angles is 90° which is the same as the angle of a right angled triangle, so measure of the angle is 90.]

7. The interior angles of a quadrilateral are together equal in measure to four right angles.
[**Hint:** Draw any one diagonal. Use Theorem 9 to prove that 2 rt $\angle$ + 2 rt $\angle$ = 4rt $\angle$ s].

REVIEW EXERCISE 10

1. Demonstrative Geometry is a branch of Mathematics. What does it do?

2. Write what you know about the following terms:

(a) Undefined Terms (b) Assumptions or Fundamental Agreements

(c) Propositions or Theorems (d) Postulates

(e) Axioms (e) Corollaries

3. Write down the four Basics of Reasoning.

4. (a) State Theorem 1 and prove it.

(b) The sum of measures of all the angles at a point in a given straight line on the same side of it is equal to two right angles. Prove.

5. (a) Prove that if two lines intersect, then the opposite vertical angles are congruent.

(b) An equilateral triangle is also equiangular, prove.

6. (a) From a point not on the line, one and only one perpendicular can be drawn to a given line.

(b) If a transversal intersects two parallel lines, the alternate angles so formed are congruent.

7. (a) The measure of an exterior angle of a triangle is equal to the sum of the measures of non-adjacent interior angles of the triangle.

(b) State Theorem 5 and prove it.

8. (a) Write 'True/False' against each statement:

(i) If two lines intersect, then vertical adjacent angles are congruent.

(ii) Through a point not on a line, one and only one line can be drawn parallel to the line.

(iii) Sum of the measures of three angles of a triangle may be equal to 180°.

(iv) If a ray is drawn through a point of a given line and forms two adjacent angles, then these angles are complementary to one another.

(v) In a right angled triangle the acute angles are complementary.

(B) Fill in the blanks with appropriate words:

(i) Three non-collinear points determine a _____.

(ii) Two distinct points determine a _____.

(iii) One and only one _____ can be drawn from a point not on the line.

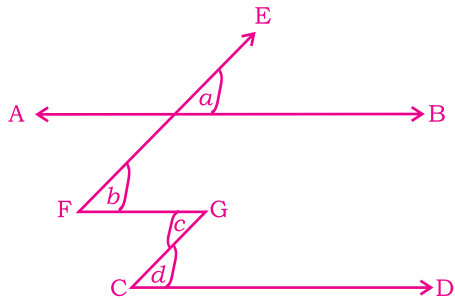
(iv) If a line cuts two parallel lines, then each pair of _____ angles and each pair of _____ angles are congruent.

(v) Measure of an exterior angle of a triangle is equal to the _____ of the measures of the non-adjacent Interior angles of the triangle.

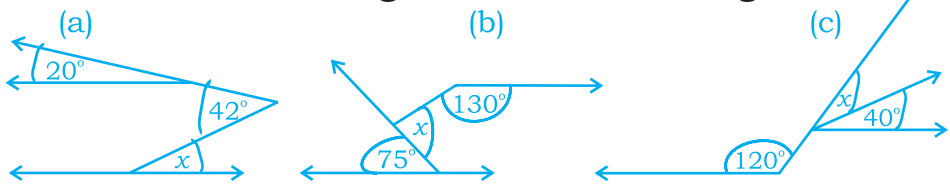
9. (i) In the figure on the right,

$m\angle a = m\angle b = m\angle c = m\angle d$

Write down four pairs of parallel lines.



(ii) In each of the figures given below, find the angle marked *x*. Numbers indicate the degree measure of the angle.



SUMMARY

Demonstrative Geometry is a branch of Mathematics in which statements concerning geometrical figures are proved through logical reasoning.

- Reasoning is the process of thinking about something in a logical way in order to form a conclusion or judgment.
- A reasoning in which we generalize specific observations into universal statements is called ***Inductive Reasoning.***

- In ***Deductive Reasoning*** we deduce particular results from general results.
- The whole framework of Deductive Reasoning is based on the four elements known as ***Basics of Reasoning*** which are:
  1. Undefined terms 2. Defined Terms
  3. Assumptions or Fundamental Agreements
  4. Propositions
- Fundamental Agreements (or Assumptions) are of two kinds.
- Axioms are those Fundamental Agreements which are related to numbers.
- Postulates are those Fundamental Agreements which are related to geometrical figures.
- Proposition is a statement proposed for discussion. Propositions are either ***Theorems*** or ***Problems*** (also called ***Riders***)
- A proposition (Theorem or Rider) consists of two parts:
  1. Hypothesis and 2. Conclusion
- A corollary is a fact which readily follows from a proved theorem.
- One theorem is said to be the converse of another theorem when the ***hypothesis*** of one is the conclusion of the other.
- **Enunciation** is the general statement of a geometrical truth which we are going to prove.
- **Given** or **Data** is the particular statement of the Hypothesis according to the figure which we have drawn.
- **To prove that** is the specific statement about the conclusion which we are going to prove.
- **Construction:** In this part, we add something to the given figure (by drawing a line, bisector etc) which may help in proving a theorem.
- **Proof:** In this last part of proving a theorem, the proposition is proved logically, putting statements in one column and its reason opposite to it in another column. The prescribed nine theorems follow this procedure.

UNIT 11

TRIGONOMETRY

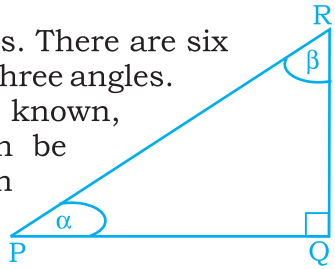
11.1 Trigonometry

11.1.1 Define Trigonometry

The word trigonometry is a Greek word. It is an important branch of Mathematics. Trigonometry deals with the relation of measures sides and angles of a triangle.

Many muslim mathematicians have made significant contribution. It plays an important role in the field of electronics, electrical engineering and many other branches of physical sciences.

Here we shall discuss right angled triangles. There are six elements of a triangle. i.e., Three sides and three angles. If the measures of any two elements are known, then measure of the other elements can be found. To find measures of the unknown elements of a right triangle, we use the knowledge of trigonometric ratios.



11.1.2 Trigonometric Ratios of Acute Angles

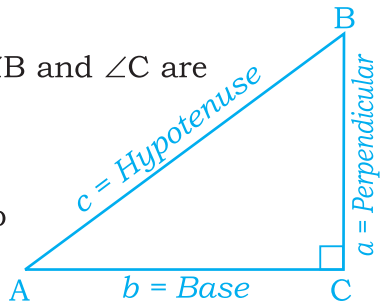
Consider a right angled triangle ACB in which $m\angle C = 90^\circ$, i.e., a right angle.

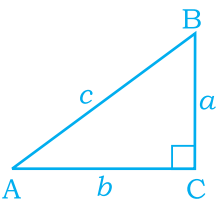
Measures of the sides opposite to ∠A, ∠B and ∠C are represented by a, b and c respectively,

i.e., $m\overline{BC} = a$, $m\overline{CA} = b$ and $m\overline{AB} = c$.

In right triangle, ratio of any of its two sides is called a trigonometric ratio.

Thus, we can form the following six possible trigonometric ratios. Consider an acute angle A of a right triangle ABC as under:





(i) Sine of $m\angle A$ = Sin $(m\angle A) = \frac{\text{Opp. Side of } m\angle A}{\text{Hypotenuse}} = \frac{m\overline{BC}}{m\overline{AB}} = \frac{a}{c}$

(ii) Cosine of $m\angle A$ = Cos $(m\angle A) = \frac{\text{Adj. Side of } m\angle A}{\text{Hypotenuse}} = \frac{m\overline{AC}}{m\overline{AB}} = \frac{b}{c}$

(iii) Tangent of $m\angle A$ = Tan $(m\angle A) = \frac{\text{Opp. Side of } m\angle A}{\text{Adjacent Side of } m\angle A} = \frac{m\overline{BC}}{m\overline{AC}} = \frac{a}{b}$

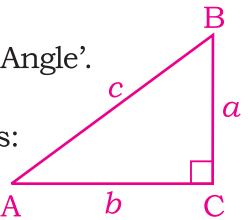
(iv) Cotangent of $m\angle A$ = Cot $(m\angle A) = \frac{\text{Adjacent Side of } m\angle A}{\text{Opposite Side of } m\angle A} = \frac{m\overline{AC}}{m\overline{BC}} = \frac{b}{a}$

(v) Secant of $m\angle A$ = Sec $(m\angle A) = \frac{\text{Hypotenuse}}{\text{Adjacent Side of } m\angle A} = \frac{m\overline{AB}}{m\overline{AC}} = \frac{c}{b}$

(vi) Cosecant of $m\angle A$ = Cosec $m\angle A = \frac{\text{Hypotenuse}}{\text{Opposite Side of } m\angle A} = \frac{m\overline{AB}}{m\overline{BC}} = \frac{c}{a}$

Note: For brevity in use;
(i) 'Side' instead of 'Measure of Side'
(ii) 'Trigonometric Ratio of Angle' instead of 'Measure of Angle'.

Activity: In the same way we can find the six Trigonometric ratios of another acute angle B as follows:



(i) Sine of $m\angle B$ = Sin $(m\angle B) = \frac{\text{Opp. Side of} \angle B}{\text{Hypotenuse}} = \frac{m\overline{AC}}{m\overline{AB}} = \text{——}$

(ii) Cosine of $m\angle B$ = Cos $(m\angle B) = \frac{\text{Adjacent Side of } m\angle B}{\text{Hypotenuse}} = \text{——} = \frac{a}{c}$

(iii) Tangent of $m\angle B$ = Tan $(m\angle B) = \frac{\text{Opp. Side of } m\angle B}{\text{——————}} = \frac{m\overline{AC}}{m\overline{BC}} = \frac{b}{a}$

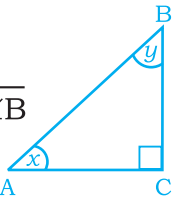
(iv) Cotangent of $m\angle B$ = Cot $(m\angle B) = \frac{\text{——————}}{\text{Opposite Side of } m\angle B} = \frac{m\overline{BC}}{m\overline{AC}} = \text{——}$

(v) Secant of $m\angle B$ = Sec $(m\angle B) = \text{——————} = \frac{m\overline{AB}}{m\overline{BC}} = \frac{c}{a}$

(vi) Cosecant of $m\angle B$ = Cosec $(m\angle B) = \frac{\text{Hypotenuse}}{\text{Opposite Side of } m\angle B} = \text{——} = \text{——}$

Note: Cosec θ is also denoted by Csc θ.

By considering the above ratios, it is observed that:



(i) $\text{Sin } m\angle A = \dfrac{1}{\text{Cosec } m\angle A}$ and $\text{Cosec } m\angle B = \dfrac{1}{\text{Sin } m\angle B}$

(ii) $\text{Cos } m\angle A = \dfrac{1}{\text{Sec } m\angle A}$ and $\text{Sec } m\angle B = \dfrac{1}{\text{Cos } m\angle B}$

(iii) $\text{Tan } m\angle A = \dfrac{1}{\text{Cot } m\angle A}$ and $\text{Cot } m\angle B = \dfrac{1}{\text{Tan } m\angle B}$

It is also observed that:

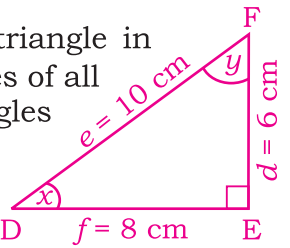
(a) $\dfrac{\text{Sin } m\angle A}{\text{Cos } m\angle A} = \text{Tan } m\angle A$ (b) $\dfrac{\text{Cos } m\angle B}{\text{Sin } m\angle B} = \text{Cot } m\angle B$

or $\dfrac{\text{Sin } x}{\text{Cos } x} = \text{Tan } x$ or $\dfrac{\text{Cos } y}{\text{Sin } y} = \text{Cot } y$

From the above results, we can get:

(iv)	$\text{Sin } m\angle A \text{ Cosec } m\angle A = 1$	or $\text{Sin } x \text{ Cosec } x = 1$
(v)	$\text{Cos } m\angle A \text{ Sec } m\angle A = 1$	or $\text{Cos } x \text{ Sec } x = 1$
(vi)	$\text{Tan } m\angle A \text{ Cot } m\angle A = 1$	or $\text{Tan } x \text{ Cot } x = 1$

Example 1. Consider DEF, a right angled triangle in which $m\angle D = x$ and $m\angle F = y$. Find the values of all the trigonometric ratios, of both the acute angles of measure x and y.



Solution:

(i) $\text{Sin } x = \dfrac{\text{Opposite Side of } m\angle D}{\text{Hypotenuse}} = \dfrac{m\overline{EF}}{m\overline{DF}} = \dfrac{d}{e} = \dfrac{6}{10} = \dfrac{3}{5}$

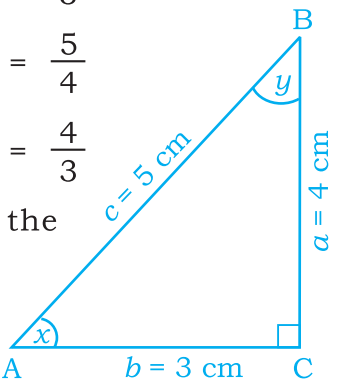
(ii) $\text{Cos } x = \dfrac{\text{Adjacent Side of } m\angle D}{\text{Hypotenuse}} = \dfrac{m\overline{DE}}{m\overline{DF}} = \dfrac{f}{e} = \dfrac{8}{10} = \dfrac{4}{5}$

(iii) $\text{Tan } x = \dfrac{\text{Opposite Side of } m\angle D}{\text{Adjacent Side}} = \dfrac{m\overline{EF}}{m\overline{DE}} = \dfrac{d}{f} = \dfrac{6}{8} = \dfrac{3}{4}$

(iv) $\text{Cosec } x = \dfrac{1}{\text{Sin } x} = \dfrac{e}{d} = \dfrac{10}{6} = \dfrac{5}{3}$

(v) $\text{Sec } x = \dfrac{1}{\text{Cos } x} = \dfrac{e}{f} = \dfrac{10}{8} = \dfrac{5}{4}$

(vi) $\text{Cot } x = \dfrac{1}{\text{Tan } x} = \dfrac{f}{d} = \dfrac{8}{6} = \dfrac{4}{3}$



Note: The students are advised to find all the trigonometric ratios of y themselves.

Example 2. In the given ΔABC, verify the following relations between trigonometric ratios.

(i) $\dfrac{\text{Cos } x}{\text{Sin } x} = \text{Cot } x$ **(ii)** $\dfrac{\text{Sin } y}{\text{Cos } y} = \text{Tan } y$ **(iii)** $\text{Tan } x \text{Cot } x = 1$

Solution: (i) $\dfrac{\text{Cos } x}{\text{Sin } x} = \text{Cot } x$

Here,

$$\text{L.H.S} = \frac{\text{Cos } x}{\text{Sin } x} = \frac{(m\overline{AC}/m\overline{AB})}{(m\overline{BC}/m\overline{AB})} = \frac{b/c}{a/c} = \frac{3/5}{4/5} = \frac{3}{\cancel{5}_1} \times \frac{\overset{1}{\cancel{5}}}{4} = \frac{3}{4} \quad \ldots\text{I}$$

$$\text{R.H.S} = \text{Cot } x = \frac{\text{Adjacent Side of } m\angle A}{\text{Opposite Side of } m\angle A} = \frac{m\,\overline{AC}}{m\,\overline{BC}} = \frac{b}{a} = \frac{3}{4} \quad \ldots\text{II}$$

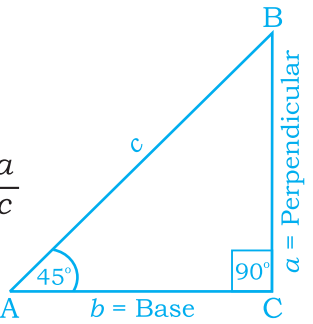
Comparing I and II, we get L.H.S = R.H.S or $\dfrac{\text{Cos } x}{\text{Sin } x} = \text{Cot } x$

(ii) $\dfrac{\text{Sin } y}{\text{Cos } y} = \text{Tan } y$

$$\text{L.H.S} = \frac{\text{Sin } y}{\text{Cos } y} = \frac{(\text{Opposite Side of } m\angle B/\text{Hypotenuse})}{(\text{Adjacent Side of } m\angle B/\text{Hypotenuse})} = \frac{(m\overline{AC}/m\overline{AB})}{(m\overline{BC}/m\overline{AB})}$$

$$= \frac{b/c}{a/c} = \frac{3/5}{4/5} = \frac{3}{\cancel{5}_1} \times \frac{\overset{1}{\cancel{5}}}{4} = \frac{3}{4} \quad \ldots\text{III}$$

Let us find the six trigonometric ratios of an acute angle of 45°.



(i) $\text{Sin } 45^\circ = \text{Sin } m\angle A = \dfrac{\text{Opp. Side of } m\angle A}{\text{Hyp}} = \dfrac{a}{c}$

$= \dfrac{b}{\sqrt{2}\,b} = \dfrac{1}{\sqrt{2}}$

Or $\text{Sin } 45^\circ = \dfrac{1}{\sqrt{2}}$

(ii) $\text{Cos } 45^\circ = \text{Cos } m\angle A = \dfrac{\text{Adj. Side of } m\angle A}{\text{Hypotenuse}} = \dfrac{b}{c} = \dfrac{b}{\sqrt{2}\,b} = \dfrac{1}{\sqrt{2}}$

or $\text{Cos } 45^\circ = \dfrac{1}{\sqrt{2}}$

(iii) $\text{Tan } 45^\circ = \text{Tan } m\angle A = \dfrac{\text{Opp. side of } m\angle A}{\text{Adj. side of } m\angle A} = \dfrac{a}{b} = 1$ or $\text{Tan } 45^\circ = 1$

(iv) $\text{Cot } 45^\circ = \text{Cot } m\angle A = \dfrac{\text{Adj. side of } m\angle A}{\text{Opp. side of } m\angle A} = \dfrac{b}{a} = 1$ or $\text{Cot } 45^\circ = 1$

(v) $\text{Sec } 45^\circ = \text{Sec } m\angle A = \dfrac{\text{Hyp}}{\text{Adjacent side of } m\angle A} = \dfrac{\sqrt{2}\,b}{b} = \sqrt{2}$

or $\text{Sec } 45^\circ = \sqrt{2}$

(vi) $\text{Cosec } m\angle A = \dfrac{\text{Hyp}}{\text{Opposite side of } m\angle A} = \dfrac{\sqrt{2}\,b}{b} = \sqrt{2}$

or $\text{Cosec } 45^\circ = \sqrt{2}$

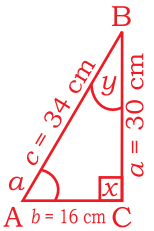
Thus

Sin 45°	Cos 45°	Tan 45°	Cot 45°	Sec 45°	Cosec 45°
$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$

(iii) $\text{Tan}\,\alpha . \text{Cot}\,\alpha = 1$ (iv) $\frac{\text{Sin}\,\beta}{\text{Cos}\,\beta} = \text{Tan}\,\beta$ (v) $\frac{\text{Cos}\,\alpha}{\text{Sin}\,\alpha} = \text{Cot}\,\alpha$

(vi) $\text{Cos}\,\alpha . \text{Cosec}\,\alpha = \text{Cot}\,\alpha$ (vii) $\frac{\text{Sec}\,\beta}{\text{Cosec}\,\beta} = \text{Tan}\,\beta$ (viii) $\frac{\text{Tan}\,\alpha}{\text{Cot}\,\alpha} = 1$

4. In the given right triangle ABC, Prove the following:



(i) $\text{Sin}\,x = \text{Cos}\,y$ (ii) $\text{Tan}\,y = \text{Cot}\,x$

(iii) $\text{Sec}\,x = \text{Cosec}\,y$ (iv) $\text{Cos}\,x = \text{Sin}\,y$

(v) $\text{Cot}\,y = \text{Tan}\,x$ (vi) $\text{Cosec}\,x = \text{Sec}\,y$

11.2 Trigonometric Ratios of Acute Angles

11.2.1 Find the values of all the trigonometric ratios of acute angles of 30°, 45°and 60° .

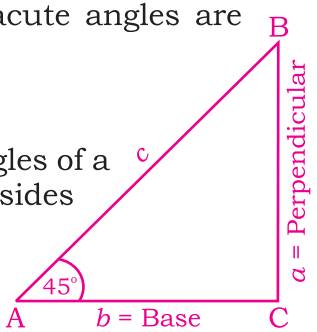
(a) Find the values of all the Trigonometric Ratios of an angle of measure of 45°.

Consider the right angled triangle ABC with right angle at point C and $m\angle A = 45^\circ$.

We know that in a right angled triangle, acute angles are complementary to each other.

So, $m\angle B = 90^\circ - m\angle A = 90^\circ - 45^\circ = 45^\circ$.

We know that if the measures of any two angles of a triangle are equal, then measures of the sides opposite to them are also equal.



i.e., $m\angle A = m\angle B = 45^\circ$, then $m\overline{BC} = m\overline{AC}$

We know that: $(\text{Hyp})^2 = (\text{Perp})^2 + (\text{Base})^2$

or $a = b$ (Pythagoras Theorem)

or $c^2 = a^2 + b^2$

or $c^2 = b^2 + b^2 = 2b^2$ $(\because\ a = b)$

$(c)^2 = (\sqrt{2}\,b)^2$, i.e., $c = \sqrt{2}\ b$.

R.H.S= Tan y $= \frac{\text{Opposite Side of } m\angle B}{\text{Adjacent Side of } m\angle B} = \frac{m\,\overline{AC}}{m\,\overline{BC}} = \frac{b}{a} = \frac{3}{4}$ IV

Comparing results III and IV, we get L.H.S. = R.H.S

(iii) Tan x Cot x = 1

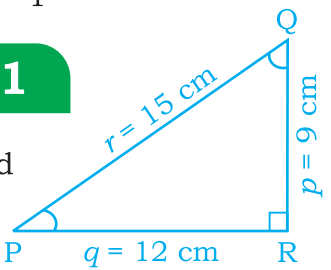
L.H.S = Tan x Cot $x = \frac{\text{Opposite side of } m\angle A}{\text{Adjacent side of } m\angle A} \times \frac{\text{Adjacent side of } m\angle A}{\text{Opposite side of } m\angle A}$

$$\frac{a}{b} \times \frac{b}{a} = \frac{\overset{1}{\cancel{4}}}{\cancel{3}_1} \times \frac{\overset{1}{\cancel{3}}}{\cancel{4}} = \frac{1}{1} = 1$$

Thus L.H.S = R.H.S. Or Tan x. Cot x = 1

EXERCISE 11.1

1. In the given right angled triangle PQR find all the trigonometric ratios for both the acute angles



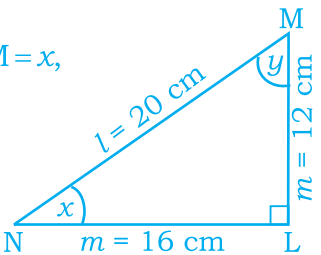
2. In the given right angled triangle LMN $m\angle M = x$, $m\angle N = y$. Find the values of the following trigonometric ratios.

(i) Sec x (ii) Cos y (iii) Tan x

(iv) Cosec y (v) Sec x (vi) Cot y

(vii) Cos x (viii) Sin y (ix) Cot x

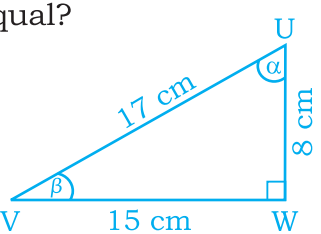
(x) Sec y (xi) Cosec x (xii) Tan y



Which of the above trigonometric ratios are equal?

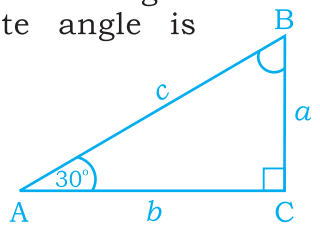
3. In the given right angled triangle UVW. Verify the following:

(i) Sin α. Cosec α = 1 (ii) Sec β. Cos β = 1



(B) To find the values of Trigonometric Ratios of an acute angle measuring 30°.

Consider the right angled triangle ABC, with right angle at point C and $m\angle A = 30°$. In given right triangle, the acute angles are complementary angles, i.e., each acute angle is complement of the other.



So, $m\angle B = 90° - m\angle A = 90° - 30° = 60°$.

In the right triangle ABC, with $m\angle A = 30°$, the measure of Hypotenuse is twice the measure of the side opposite to $\angle A$

i.e., $c = 2a$.

By Pythagoras Theorem, $c^2 = a^2 + b^2$

Putting $c = 2a$, we get: $(2a)^2 = a^2 + b^2$

or $4a^2 - a^2 = b^2$ or $3a^2 = b^2$ So, $b = \sqrt{3}\,a$

(i) $\text{Sin } 30° = \dfrac{\text{Opposite Side of } m\angle A}{\text{Hypotenuse}} = \dfrac{m\overline{BC}}{m\overline{AB}} = \dfrac{a}{c} = \dfrac{a}{2a} = \dfrac{1}{2}$

(ii) $\text{Cos } 30° = \dfrac{\text{Adjacent Side of } m\angle A}{\text{Hypotenuse}} = \dfrac{m\overline{AC}}{m\overline{AB}} = \dfrac{b}{c} = \dfrac{\sqrt{3}\,a}{2a} = \dfrac{\sqrt{3}}{2}$

(iii) $\text{Tan } 30° = \dfrac{\text{Opposite Side of } m\angle A}{\text{Adjacent Side of } m\angle A} = \dfrac{m\overline{BC}}{m\overline{AC}} = \dfrac{a}{b} = \dfrac{a}{\sqrt{3}\,a} = \dfrac{1}{\sqrt{3}}$

(iv) $\text{Cot } 30° = \dfrac{\text{Adjacent Side of } m\angle A}{\text{Hypotenuse}} = \dfrac{m\overline{AC}}{m\overline{BC}} = \dfrac{b}{a} = \dfrac{\sqrt{3}\,a}{a} = \sqrt{3}$

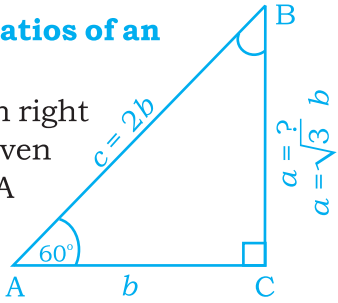
(v) $\text{Sec } 30° = \dfrac{\text{Hypotenuse}}{\text{Adjacent Side of } m\angle A} = \dfrac{m\overline{AB}}{m\overline{AC}} = \dfrac{c}{b} = \dfrac{2\,a}{\sqrt{3}\,a} = \dfrac{2}{\sqrt{3}}$

(vi) $\text{Cosec } 30° = \dfrac{\text{Hypotenuse}}{\text{Opposite Side of } m\angle A} = \dfrac{m\overline{AB}}{m\overline{BC}} = \dfrac{c}{a} = \dfrac{2\,a}{a} = 2$

Thus

Sin 30°	Cos 30°	Tan 30°	Cot 30°	Sec 30°	Cosec 30°
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2

(c) To find the values of Trigonometric Ratios of an acute angle measuring 60°.



Consider the right angled triangle ABC, with right angle at point C and $m\angle A = 60^\circ$. In the given right angled triangle ABC, $m\angle B = 90^\circ - m\angle A = 90^\circ - 60^\circ = 30^\circ$. Thus, $m\angle A = 60^\circ$, $m\angle B = 30^\circ$ and $m\angle C = 90^\circ$,

Hence in the given right angled triangle ABC with $m\angle B = 30^\circ$, the measure of the Hypotenuse is twice the measure of the side opposite to it, i.e., $c = 2b$.

(By Pythagoras Theorem)

$(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{Perpendicular})^2$

$$c^2 = b^2 + a^2$$

or $(2b)^2 = b^2 + a^2$ (putting $c = 2b$)

or $4b^2 - b^2 = a^2$

$\Rightarrow$ $a^2 = 3b^2$ or $\sqrt{a^2} = \sqrt{3b^2}$

or $a = \sqrt{3}\ b$

(i) $\text{Sin } m\angle A = \text{Sin } 60^\circ = \dfrac{\text{Opposite Side of } m\angle A}{\text{Hypotenuse}} = \dfrac{a}{c} = \dfrac{\sqrt{3}\ b}{2b} = \dfrac{\sqrt{3}}{2}$

(ii) $\text{Cos } m\angle A = \text{Cos } 60^\circ = \dfrac{\text{Adjacent Side of } m\angle A}{\text{Hypotenuse}} = \dfrac{b}{2b} = \dfrac{1}{2}$

(iii) $\text{Tan } m\angle A = \text{Tan } 60^\circ = \dfrac{\text{Opposite Side of } m\angle A}{\text{Adjacent Side of } m\angle A} = \dfrac{a}{b} = \dfrac{\sqrt{3}\ b}{b} = \sqrt{3}$

(iv) $\text{Cot } m\angle A = \text{Cot } 60^\circ = \dfrac{\text{Adjacent Side of } m\angle A}{\text{Opposite Side of } m\angle A} = \dfrac{b}{a} = \dfrac{b}{\sqrt{3}\ b} = \dfrac{1}{\sqrt{3}}$

(v) $\text{Sec } m\angle A = \text{Sec } 60^\circ = \dfrac{\text{Hypotenuse}}{\text{Adjacent Side of } m\angle A} = \dfrac{c}{b} = \dfrac{2b}{b} = 2$

(vi) $\text{Cosec } m\angle A = \text{Cosec } 60^\circ = \dfrac{\text{Hypotenuse}}{\text{Opp. Side of } m\angle A} = \dfrac{c}{a} = \dfrac{2\ b}{\sqrt{3}\ b} = \dfrac{2}{\sqrt{3}}$

Thus, the six trigonometric ratios for an angle measuring 60° are:

Sin 60°	Cos 60°	Tan 60°	Cot 60°	Sec 60°	Cosec 60°
$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$

Note: Whatever be the size of a side in a right triangle, the trigonometric ratios of an acute angle is always the same.

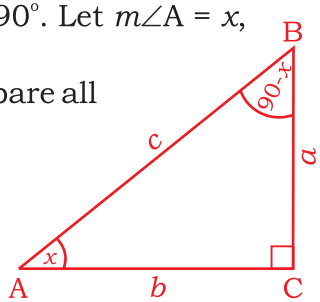
The results derived above can be written in the form of a table:

Angle θ	Sin θ	Cos θ	Tan θ	Cot θ	Sec θ	Cosec θ
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$

11.2.4 Define Trigonometric Ratios of Complementary Angles

We have already learnt that the acute angles of a right angled triangle are complementary angles. ABC is a right angled triangle, with $m\angle C = 90^\circ$, $m\angle A + m\angle B = 90^\circ$. Let $m\angle A = x$, then $m\angle B = 90^\circ - m\angle A$ or $m\angle B = 90^\circ - x$.

Consider right angled triangle ABC and compare all the six trigonometric ratios of $m\angle A = x$ and of $m\angle B = (90^\circ - x)$ which are complements of one another.



If $x = 30^\circ$, then angles of meaure 30° and $(90 - 30)^\circ$ or of 30° and 60° are complementary angles.

From the above figure, we see that:

$$\text{(i) Sin } x = \frac{a}{c} = \text{Cos } (90 - x) \Rightarrow \text{Sin } x = \text{Cos } (90 - x)$$

(ii) $\text{Cos } x = \frac{b}{c} = \text{Sin } (90 - x) \Rightarrow \text{Cos } x = \text{Sin } (90 - x)$

Also from the above table we see that:

$$\text{Sin } 30^\circ = \frac{1}{2} = \text{Cos } (90 - 30)^\circ \Rightarrow \text{Sin } 30^\circ = \text{Cos } 60^\circ$$

$$\text{And Cos } 30^\circ = \frac{\sqrt{3}}{2} = \text{Sin } (90 - 30)^\circ \Rightarrow \text{Cos } 30^\circ = \text{Sin } 60^\circ$$

From the above discussion, we conclude that:

(a) Sin and Cos of complementary angles are equal.

(b) Tan and Cot of complementary angles are equal.

Examples. $\text{Tan } 30^\circ = \frac{1}{\sqrt{3}} = \text{Cot } (90-30)^\circ \Rightarrow \text{Tan } 30^\circ = \text{Cot } 60^\circ$

$$\text{Cot } 30^\circ = \sqrt{3} = \text{Tan } (90-30)^\circ \Rightarrow \text{Cot } 30^\circ = \text{Tan } 60^\circ.$$

(c) Sec and Cosec of complementary angles are equal.

Examples. $\text{Sec } x = \text{Cosec } (90-x)$ and $\text{Cosec } x = \text{Sec } (90-x)$.

$$\text{Sec } 30^\circ = \frac{2}{\sqrt{3}} = \text{Cosec } (90-30)^\circ \Rightarrow \text{Sec } 30^\circ = \text{Cosec } 60^\circ.$$

$$\text{Sec } 60^\circ = 2 = \text{Cosec } (90-60)^\circ \Rightarrow \text{Sec } 60^\circ = \text{Cosec } 30^\circ.$$

Hence,

$\text{Sin } 30^\circ = \frac{1}{2} = \text{Cos } 60^\circ$

$\text{Cos } 30^\circ = \frac{\sqrt{3}}{2} = \text{Sin } 60^\circ$

$\text{Tan } 30^\circ = \frac{1}{\sqrt{3}} = \text{Cot } 60^\circ$

$\text{Cot } 30^\circ = \sqrt{3} = \text{Tan } 60^\circ$

$\text{Sec } 30^\circ = \frac{2}{\sqrt{3}} = \text{Cosec } 60^\circ$

$\text{Cosec } 30^\circ = 2 = \text{Sec } 60^\circ$

Similarly we can say:

$\text{Sin } 40^\circ = \text{Cos } 50^\circ$

$\text{Cos } 70^\circ = \text{Sin } 20^\circ$

$\text{Tan } 10^\circ = \text{Cot } 80^\circ$

$\text{Cot } 20^\circ = \text{Tan } 70^\circ$

$\text{Sec } 50^\circ = \text{Cosec } 40^\circ$

$\text{Cosec } 20^\circ = \text{Sec } 70^\circ$

Example. Find the value of the following:

(i) $\text{Sin } 30^\circ \times \text{Cos } 45^\circ + \text{Cos } 60^\circ \times \text{Sin } 45^\circ$

(ii) $\text{Tan } 30^\circ \times \text{Cot } 60^\circ + \text{Cot } 30^\circ \times \text{Tan } 60^\circ$

(iii) $\text{Sec } 30^\circ \times \text{Cosec } 45^\circ + \text{Sec } 45^\circ \times \text{Cosec } 60^\circ$

Solution:

(i) $\text{Sin } 30^\circ \times \text{Cos } 45^\circ + \text{Cos } 60^\circ \times \text{Sin } 45^\circ$

Putting the values from the table in the expression given above; we get

$$\text{Sin } 30^\circ \times \text{Cos } 45^\circ + \text{Cos } 60^\circ \times \text{Sin } 45^\circ$$

$$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{1+1}{2\sqrt{2}} = \frac{\cancel{2}}{\cancel{2}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(ii) $\text{Tan } 30^\circ \times \text{Cot } 60^\circ + \text{Cot } 30^\circ \times \text{Tan } 60^\circ$

$\text{Tan } 30^\circ = \frac{1}{\sqrt{3}}$, $\text{Tan } 60^\circ = \sqrt{3}$, $\text{Cot } 30^\circ = \sqrt{3}$

and $\text{Cot } 60^\circ = \frac{1}{\sqrt{3}}$ in the given expression,

$$\text{Tan } 30^\circ \times \text{Cot } 60^\circ + \text{Cot } 30^\circ \times \text{Tan } 60^\circ$$

$$= \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} + \sqrt{3} \times \sqrt{3} = \frac{1}{\sqrt{3 \times 3}} + \sqrt{3 \times 3}$$

$$= \frac{1}{3} + \frac{3}{1} = \frac{1+9}{3} = \frac{10}{3} = 3\frac{1}{3} = 3.333$$

(iii) $\text{Sec } 30^\circ \times \text{Cosec } 45^\circ + \text{Sec } 45^\circ \times \text{Cosec } 60^\circ$

We know $\text{Sec } 30^\circ = \frac{2}{\sqrt{3}} = \text{Cosec } 60^\circ$ and $\text{Cosec } 45^\circ = \sqrt{2} = \text{Sec } 45^\circ$.

So, $\text{Sec } 30^\circ \times \text{Cosec } 45^\circ + \text{Sec } 45^\circ \times \text{Cosec } 60^\circ$

$$= \frac{2}{\sqrt{3}} \times \frac{2}{\sqrt{3}} + \frac{\sqrt{2} \times \sqrt{2}}{1} = \frac{2 \times 2}{\sqrt{3 \times 3}} + \frac{\sqrt{2 \times 2}}{1}$$

$$= \frac{4}{3} + \frac{2}{1} = \frac{4+6}{3} = \frac{10}{3} = 3\frac{1}{3} = 3.333$$

Example 2. Find: **(a)** Value of x when the value of $\text{Sin } x$ is $\frac{1}{2}$

(b) Value of y when the value of $\text{Tan } y = 1$

Solution: (a) We know that: $\text{Sin } 30^\circ = \frac{1}{2}$ (from the table)

But here $\text{Sin } x = \frac{1}{2}$

This implies that: $\text{Sin } x = \text{Sin } 30^\circ \Rightarrow x = 30^\circ$.

(b) We know that, $\text{Tan } 45^\circ = 1$ (from the table)

But here $\text{Tan } y = 1 \Rightarrow \text{Tan } y = \text{Tan } 45^\circ \Rightarrow y = 45^\circ$.

EXERCISE 11.2

1. Find the value of x, when the value of Cos x is:

(i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{1}{\sqrt{2}}$ (iii) $\frac{1}{2}$

2. Find the value of y, when the value of Tan y is:

(i) $\sqrt{3}$ (ii) 1 (iii) $\frac{1}{\sqrt{3}}$

3. Find the value of $m\angle A$, when the value of Cosec $m\angle A$ is:

(i) $\frac{2}{1}$ (ii) $\sqrt{2}$ (iii) $\frac{2}{\sqrt{3}}$

4. Find the value of the following:

(i) $\text{Sin } 30^\circ \times \text{Sin } 60^\circ + \text{Cos } 30^\circ \times \text{Cos } 60^\circ$

(ii) $\text{Sec } 30^\circ \times \text{Cosec } 60^\circ + \text{Tan } 60^\circ \times \text{Cot } 30^\circ$

(iii) $\text{Cos } 30^\circ \times \text{Sin } 30^\circ + \text{Cosec } 30^\circ \times \text{Sec } 30^\circ$

(iv) $\text{Sin } 60^\circ \times \text{Sin } 45^\circ - \text{Cos } 30^\circ \times \text{Cos } 45^\circ$

(v) $\text{Tan } 30^\circ \times \text{Cot } 60^\circ - \text{Sin } 45^\circ \times \text{Cos } 45^\circ$

(vi) $\text{Sec } 45^\circ \times \text{Cosec } 45^\circ + \text{Sec } 30^\circ \times \text{Cosec } 60^\circ$

(vii) $\text{Cot } 60^\circ \times \text{Sec } 30^\circ - \text{Cosec } 60^\circ \times \text{Tan } 30^\circ$

5. Find the value of x in each of the following cases:

(i) $\text{Sin } 30^\circ = \text{Cos } x$ (ii) $\text{Tan } 60^\circ = \text{Cot } x$ (iii) $\text{Sec } x = \text{Cosec } 45^\circ$

(iv) $\text{Cos } x = \text{Sin } 70^\circ$ (v) $\text{Sec } 40^\circ = \text{Cosec } x$ (vi) $\text{Cot } x = \text{Tan } 55^\circ$

(vii) $\text{Cosec } 10^\circ = \text{Sec } x$ (viii) $\text{Tan } x = \text{Cot } 65^\circ$ (ix) $\text{Sin } x = \text{Cos } 35^\circ$

6. Fill in the blanks:

(i) $\text{Sin } 35^\circ = \text{Cos } \_\_\_\_$ (ii) $\text{Tan } 65^\circ = \text{Cot } \_\_\_\_$ (iii) $\text{Sec } 20^\circ = \text{Cosec } \_\_\_\_$

(iv) $\text{Cot } 53^\circ = \text{Tan } \_\_\_\_$ (v) $\text{Cos } 56^\circ = \text{Sin } \_\_\_\_$ (vi) $\text{Cosec } 75^\circ = \text{Sec } \_\_\_\_$

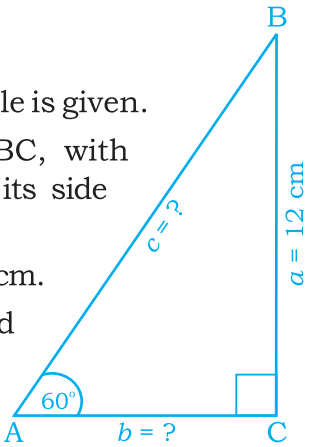
11.2.5 Solve Right Angled Triangles using Trigonometric Ratios

We have already learnt that a triangle has six elements. If the measures of three of these elements, including at least one side are known, then measure of the other three elements of the triangle can be found. The process of finding the unknown elements of a triangle is known as **solving the triangle.**

Here we have to solve only right triangles.

Case I: Measure of one side and one acute angle is given.

Example 1. Solve right angled triangle ABC, with $m\angle C = 90^\circ$, $m\angle A = 60^\circ$ and the measure of its side opposite to $m\angle A = 12$ cm.



Solution: Here $m\angle C = 90^\circ$, $m\angle A = 60^\circ$ and $a = 12$cm.

Now we have to find $m\angle B$, $m\overline{AC} = b$ and $m\overline{AB} = c$

We also know that in a right angled triangle, acute angles are complementary.

Therefore, $m\angle A + m\angle B = 90^\circ$

$$m\angle B = 90^\circ - 60^\circ = 30^\circ$$

$$\text{Tan } m\angle A = \text{Tan } 60^\circ = \frac{\text{Opposite Side of } m\angle A}{\text{Adjacent Side of } m\angle A} = \frac{a}{b} = \frac{12}{b}$$

or $\sqrt{3} = \frac{12}{b}$ or $b = \frac{12}{\sqrt{3}} = 4\sqrt{3} = 4 \times 1.732 = 6.928$cm

Again, $\text{Sin } m\angle A = \text{Sin } 60^\circ = \frac{a}{c} = \frac{12}{c}$

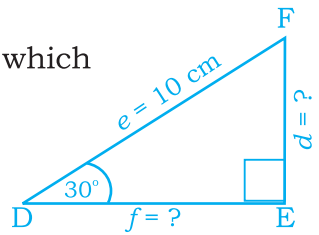
$\Rightarrow \frac{\sqrt{3}}{2} = \frac{12}{c} \quad \Rightarrow c\sqrt{3} = 12 \times 2$

$$\Rightarrow c = \frac{24}{\sqrt{3}} = \frac{24\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\overset{1}{\cancel{24}}\sqrt{3}}{\underset{1}{\cancel{3}}} = 8\sqrt{3}$$

Thus, $m\angle B = 30^\circ$, $m\overline{AC} = 4\sqrt{3} = 6.9$ cm, $m\overline{AB} = 13.8$ cm

Case II: Measure of Hypotenuse and one acute angle is given.

Example 2. Solve a right triangle DEF, when $m\angle E = 90^\circ$, $m\angle D = 30^\circ$ and $m\overline{DF}$ (Hypotenuse) = 10 cm.



Solution: Here DEF is a right triangle, in which $m\angle E = 90^\circ$, $m\angle D = 30^\circ$ and

Hypotenuse = $m\overline{DF} = 10$ cm

We have to find $m\angle F$, $m\overline{DE} = f$, $m\overline{EF} = d$

We know that:

$$\text{Sin } m\angle D = \text{Sin } 30^\circ = \frac{\text{Opposite Side of } m\angle D}{\text{Hypotenuse}} = \frac{d}{e} = \frac{d}{10}$$

or $\quad \text{Sin } 30^\circ = \frac{d}{10}$

or $\quad \frac{1}{2} = \frac{d}{10}$, or $d = 5$cm

Also $\quad \frac{f}{e} = \text{Cos } m\angle D \Rightarrow \frac{f}{e} = \text{Cos } 30^\circ$

$$\Rightarrow \quad \frac{f}{10} = \frac{\sqrt{3}}{2} \quad \Rightarrow f = \frac{10\sqrt{3}}{2} = 5\sqrt{3} = 8.7 \text{ cm}$$

$m\angle D + m\angle F = 90^\circ$ (Complementary angles)

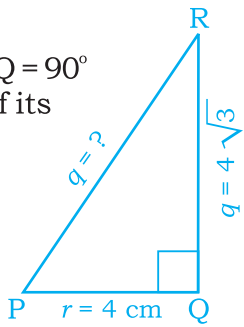
So, $\quad m\angle F = 90^\circ - m\angle D = 90^\circ - 30^\circ = 60^\circ$

Hence, the solution of the given ΔDEF is as under:

$m\angle F = 60^\circ$, $m\overline{DE} = 5\sqrt{3} = 8.7$ cm and, $m\overline{EF} = 5$ cm

Case III: The measure of two sides is given.

Example 3. Solve a right angled triangle PQR with $m\angle Q = 90^\circ$ and measure of its perpendicular is $4\sqrt{3}$cm and of its base is 4cm.



Solution: Here ΔPQR is a right angled triangle, in which $m\angle Q = 90^\circ$, $m\overline{PQ} = 4$cm, $m\overline{QR} = 4\sqrt{3}$ cm.

Now we have to find $m\angle P$, $m\angle R$ and

$m\overline{PR}$ = Hypothenuse.

According to Pythagoras Theorem:

$(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{Perpendicular})^2$

$q^2 = r^2 + p^2$

$\Rightarrow (q)^2 = (4)^2 + (4\sqrt{3})^2 = 16 + 48 = 64$

$\therefore$ Hypotenuse $= q = \sqrt{64} = 8\text{cm}$. So, $m\overline{PR} = 8$ cm.

We know that $\text{Tan}\, m\angle P = \dfrac{\text{Opposite Side of } m\angle P}{\text{Adjacent Side}} = \dfrac{4\sqrt{3}}{4} = \sqrt{3}$.

$\therefore\ m\angle P = 60^\circ$

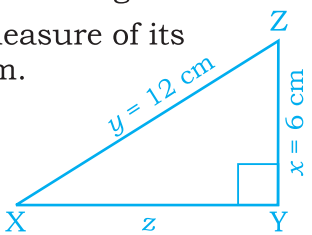
Then $m\angle R = 90^\circ - 60^\circ = 30^\circ$.

Hence solution of the given triangle is as under:

$m\angle P = 60^\circ$, $m\angle R = 30^\circ$ and Hyp $= 8$ cm.

Case IV: The measures of one side and Hypotenuse is given.

Example 4. Solve ΔXYZ when $m\angle Y = 90^\circ$, measure of its Perpendicular is 6cm and Hypotenuse is 12 cm.



Solution:

Here we have to find the unknown elements.

$m\angle X = ?$, $m\angle Z = ?$ and $m\overline{XY} = z = ?$

We know that $\text{Sin}\, m\angle X = \dfrac{\text{Opposite Side of } m\angle X}{\text{Hypotenuse}} = \dfrac{m\overline{YZ}}{m\overline{XZ}} = \dfrac{6}{12} = \dfrac{1}{2}$.

Thus, $\text{Sin}\, m\angle X = \dfrac{1}{2} \Rightarrow m\angle X = 30^\circ$.

Now $m\angle Z = 90^\circ - m\angle X = 90^\circ - 30^\circ = 60^\circ$.

Now: $\dfrac{z}{y} = \text{Cos}\, m\angle X \Rightarrow \dfrac{z}{12} = \text{Cos}\, 30^\circ = \dfrac{\sqrt{3}}{2}$

$\Rightarrow z = \dfrac{\sqrt{3}}{2} \times 12 = 6\sqrt{3}$

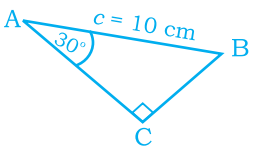
Hence, $m\angle X = 30^\circ$, $m\angle Z = 60^\circ$ and $m\overline{XY} = 6\sqrt{3}$ cm.

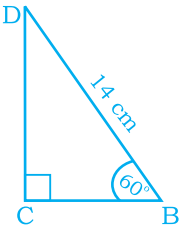
EXERCISE 11.3

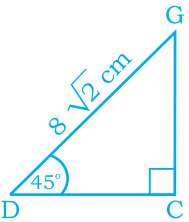
A. Solve the following right angled triangles:

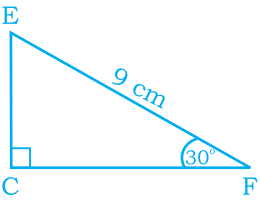
1. ΔABC, $m\angle C = 90^\circ$, $m\angle A = 30^\circ$ and Perpendicular $m\overline{BC} = 3$cm
2. ΔDEF, $m\angle E = 90^\circ$, $m\angle D = 60^\circ$ and Base $m\overline{DE} = 4$cm
3. ΔLMN, $m\angle M = 90^\circ$, $m\angle N = 45^\circ$ and Perpendicular $m\overline{ML} = 8$cm

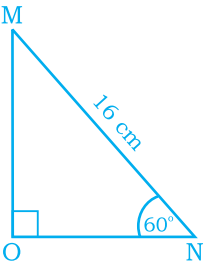
B. Find unknown elements of the following right angled triangles.

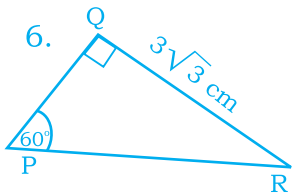
1. 

2. 

3. 

4. 

5. 

6. 

C. Solve the following right angled triangles.

1. ΔLMN, $m\angle M = 90^\circ$, Perpendicular side $l = 4$cm and Base side $n = 4\sqrt{3}$
2. ΔPQR, $m\angle Q = 90^\circ$, Perpendicular $p = 6\sqrt{3}$ cm, and base side $r = 6$cm
3. ΔXYZ, $m\angle Y = 90^\circ$, Perpendicular side $x = 4$ cm and Base side $z = 7$ cm.

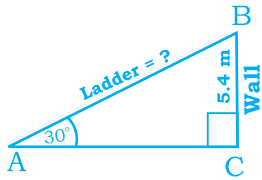
D. Solve the following right angled triangles.

1. ΔACB, $m\angle C = 90^\circ$, $a = 5$cm and Hypotenuse $c = 10$cm
2. ΔDEF, $m\angle C = 90^\circ$, $d = 7\sqrt{3}$ cm and Hypotenuse $f = 14$cm
3. ΔLMN, $m\angle L = 90^\circ$, $m = 8$cm and Hypotenuse $l = 8\sqrt{2}$ cm

11.2.6 To Solve Real Life Problems of Heights and Distances

The problems of finding heights and distances of objects in fact, is the same as that of solving right angled triangles. In this case we must draw the triangle. Sometimes, the triangle is a section of the diagram that accompanies the problem. However, we must redraw the triangle. This will avoid any confusion and help us not only to identify correctly the sides but also to choose the correct ratios.

Example 1. A window cleaner has a ladder. He leans it against a wall which makes an angle of 30° with the floor. It reaches the height of 5.4m on the wall. What is the length of the ladder?



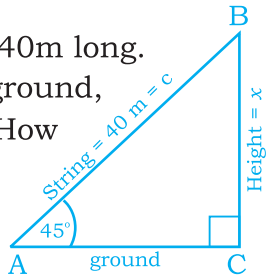
Solution: First we draw the situation as a right triangle.

Now $\frac{5.4}{c} = \text{Sin } m\angle A \Rightarrow \frac{5.4}{c} = \text{Sin } 30^\circ \Rightarrow \frac{5.4}{c} = \frac{1}{2}$

$\Rightarrow c \times 1 = 2 \times 5.4 \Rightarrow c = 10.8\text{m}$

Hence, length of the ladder is 10.8m.

Example 2. A boy is flying a kite with a string 40m long. The string is being held at 1m above the ground, making an angle of 45° with the horizontal. How high is the kite?



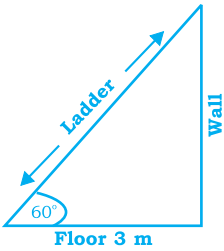
Solution: First we draw the figure which is a right triangle. Let x be the height of the kite.

Then $\frac{x}{c} = \frac{\text{Opposite Side of } m\angle A}{\text{Hypotenuse}} \Rightarrow \frac{x}{c} = \text{Sin } 45^\circ \Rightarrow \frac{x}{40} = \frac{1}{\sqrt{2}}$

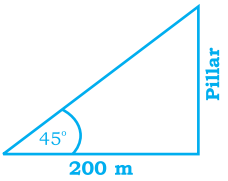
$\Rightarrow x = \frac{1}{\sqrt{2}} \times 40 = 0.707 \times 40 \Rightarrow x = 28.28 = 28.3 \text{ m (approx)}.$

Thus, kite is at a height of 28.3 m + 1 m = 29.3 m (approx) from the ground.

EXERCISE 11.3

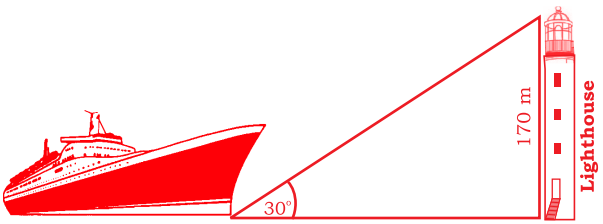


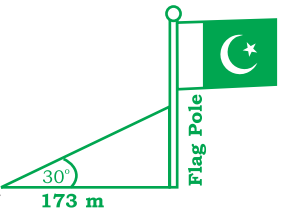
1. A ladder rests against a wall as shown in the figure. It makes an angle of 60° with the floor. If foot of the ladder is 3m away from the wall. What is the length of ladder?



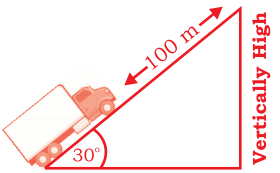
2. The foot of a pillar is at a distance of 200m from a point on the ground as shown in the figure. The angle which the pillar makes from this point is of 45°. Find the height of the pillar.

3. On the bank of a sea, there is a light house 170m high. A ship standing in the sea, makes an angle of 30° with the top of the light house, as shown in the figure. Find the distance between foot of the light house and the ship.

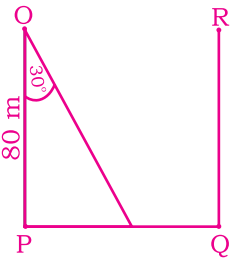




4. There is a flag-pole as shown in the figure. The angle of the top of the flag-pole from a point on the ground 20 metres away from the foot of the pole is 30°. Find the height of the pole.



5. A truck drives up a 100m incline. The angle that the truck drives up is of 30° as shown in the figure. How high did the truck rise vertically?



6. Two trees at points P and Q are on a side of a road. Tree $\overline{PO}$ is perpendicular on the road $\overline{PQ}$ as shown in the figure. Height $\overline{PO}$ is measured to be 80m. The angle POQ is measured as 30°. What is the horizontal distance between the trees?

REVIEW EXERCISE 11

1. Show that:

(i) $2 \sin 45^\circ + \frac{1}{2} \operatorname{Cosec} 45^\circ = \frac{3}{\sqrt{2}}$

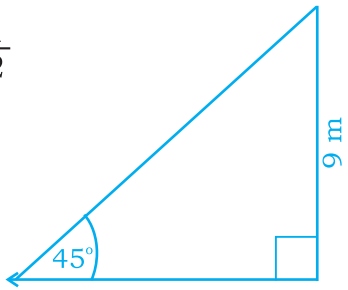
(ii) $\cot 60^\circ . \cos 30^\circ - \sin 60^\circ \sin 30^\circ = 0$

(iii) $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ = \frac{1}{2}$

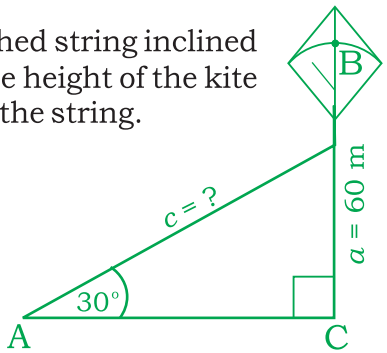
(iv) $(\cos 30^\circ - \sin 30^\circ) \times (\cos 30^\circ + \sin 30^\circ) = \frac{1}{2}$

(v) $\sin 45^\circ \times \cos 45^\circ + \sin 30^\circ \times \cos 60^\circ = \frac{3}{4}$

2. A tree having 9m height is on one bank of a river. It makes an angle of 45° on the opposite side of the river; as shown in the figure. Find the width of the river.



3. A girl flying a kite attached it to a stretched string inclined at an angle of 30° with the horizontal. If the height of the kite from the ground is 60m, find the length of the string.



4. Fill in the blanks:

(i) $\operatorname{Cosec} 30^\circ =$ _____

(ii) $\cot 60^\circ =$ _____

(iii) $\cos 45^\circ =$ _____

(iv) $\sec 30^\circ =$ _____

(v) Reciprocal of $\tan m\angle C =$ _____

(vi) $\sin m\angle A = \cos$ _____

(vii) $\cos (90 - m\angle A) = \sin$ _____

(viii) $\tan 30^\circ = \cot$ _____

(ix) $\operatorname{Cosec} 10^\circ = \sin$ _____

(x) $\sec (90 - \theta) = \operatorname{Cosec}$ _____

5. Encircle the correct answer.

(i) The value of Cos 60° is _____

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$

(ii) The value of Cot 30° = _____

(a) $\frac{1}{\sqrt{3}}$ (b) 1 (c) $\sqrt{3}$

(iii) The value of Tan 45° = _____

(a) 2 (b) $\sqrt{2}$ (c) 1

(iv) The value of Cos^2 30° = _____

(a) $\frac{3}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{2}}$

(v) The value of Cosec 60° = _____

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{1}{2}$

(vi) The value of Sin^2 45° = _____

(a) $\frac{1}{\sqrt{2}}$ (b) 2 (c) $\frac{1}{2}$

(vii) The value of Cosec^2 60° = _____

(a) 2 (b) $\frac{4}{3}$ (c) $\frac{2}{\sqrt{3}}$

6. Write T for true and F for false.

(i) $\text{Cos}\,\theta = \frac{1}{\text{Sin}\,\theta}$ (ii) $\text{Cosec}\,\theta = \frac{1}{\text{Sec}\,\theta}$ (iii) $\sec\beta = \frac{1}{\text{Cosec}\,\beta}$

(iv) $\text{Tan}\,\theta . \text{Cot}\,\theta = 1$ (v) $\text{Tan}\,x . \text{Cot}\,x = 1$ (vi) $\frac{\text{Adj. Side of } m\angle A}{\text{Hypotenuse}} = \text{Sin}\, m\angle A$

(vii) $\text{Cot}\,x = \frac{\text{Cos}\,x}{\text{Sin}\,x}$ (viii) $\text{Sin}\,20° = \text{Cosec}\,70°$

(ix) The inverse of $\text{Sin}\,x = \text{Cosec}\,x$ (x) $\text{Cos}\,(90° - 30°) = \text{Sec}\,60°$

SUMMARY

- Trigonometry is the relation of measurement of elements of a triangle.
- There are six trigonometric ratios; Viz, Sine, Cosine, Tangent, Cotangent, Secant and Cosecant.
- Trigonometric ratios are used to relate the measures of angles to the lengths of the sides of a right triangle.
- Three most common ratios in trigonometry are:

(i) $\text{Sin}\,\theta = \dfrac{\text{Opposite Side of given angle}}{\text{Hypotenuse}}$

(ii) $\text{Cos}\,\theta = \dfrac{\text{Adjacent Side of given angle}}{\text{Hypotenuse}}$

(iii) $\text{Tan}\,\theta = \dfrac{\text{Opposite Side of given angle}}{\text{Adjacent Side of given angle}}$

- $\text{Sin}\,\theta = \dfrac{1}{\text{Cosec}\,\theta}$, $\text{Cos}\,\theta = \dfrac{1}{\text{Sec}\,\theta}$, $\text{Tan}\,\theta = \dfrac{1}{\text{Cot}\,\theta}$
- $\text{Tan}\,(90^\circ - \theta) = \text{Cot}\,\theta$, $\text{Cot}\,(90^\circ - \theta) = \text{Tan}\,\theta$
- $\text{Sec}\,(90^\circ - \theta) = \text{Cosec}\,\theta$, $\text{Cosec}\,(90^\circ - \theta) = \text{Sec}\,\theta$

Table for values of Trigonometric ratios of angles of different measures.

θ	Sin	Cos	Tan	Cot	Sec	Cosec
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$

UNIT 12

INFORMATION HANDLING

Information handling is a part of Statistics.

Statistics is the branch of Mathematics concerned with collecting, classifying, analyzing and interpreting data .

In the Previous classes we have learnt the concepts of bar graph, frequency, class limits, class boundaries, range and size of the class interval.

Now we will learn some more concepts but first revise the concepts learnt in previous class in the form of an exercise.

REVISION EXERCISE

1. Find the frequency, class limits, class boundaries, class interval, upper class limit and size of each class from the following table.

Marks obtained	24-28	29-33	34-38	39-43	44-48
Number of students	03	16	12	23	16

2. 30 students of a class appeared in a Maths test. Out of 10 marks, the obtained marks by the students are given below:

4	3	3	0	2	2	4	3	3	5
2	2	4	3	5	1	0	3	5	4
3	4	4	0	3	2	3	6	0	1

(i) Form a frequency distribution table from the given information.

(ii) What is the range?

(iii) Which number of marks are obtained by maximum number of students?

(iv) Which number of marks are obtained by minimum number of students?

3. The staff members of an office were asked, “How many hours daily they pass with their children”. The answers given in hours were as shown in the table under follows:

4	3	1	6	2	2	3	1	4
5	3	4	1	2	5	3	2	2
3	2	2	3	1	1	4	2	3

(i) Form a frequency table from the given information.
(ii) What is the number of participants in this survey?
(iii) How many persons passed the maximum time with the children?
(iv) How many hours do the maximum number of persons passed time with the children?

4. A test of 150 persons of an institution was conducted for 10 marks. The score is given in the following table:

Score (Marks Obtained)	1	2	3	4	5	6	7	8	9	10
Frequency	6	12	15	21	35	24	20	10	6	1

(i) What is the number of persons obtaining highest number of marks?
(ii) What is the number of persons obtaining lowest number of marks?
(iii) What is the number of marks obtained by maximum number of persons?

12.1 Frequency Distribution

12.1.1 Define Frequency, Frequency Distribution

(a) Frequency In the raw data, many observations (values or numerical facts) recur again and again. The number of times a value occurs in the data is called its frequency, usually denoted by 'f'.

(b) Frequency Distribution While summarizing a large raw-data, it is often useful to distribute the data into classes or categories and to determine the number of individuals belonging to each class called the class-frequency.

A tabular arrangement of data together with class frequencies is called a Frequency Distribution or Frequency Table.

12.1.2 Construction of a Frequency Table

In order to construct a Frequency table, we proceed as under:

Step 1: Find the range of the raw-data and determine the class size.

$$\text{Size of the class interval} = \frac{\text{Largest value} - \text{Smallest value}}{\text{Number of classes}}$$

Step 2: Write down the class -Intervals of equal size.

Step 3: Draw an outline of table having columns as under.

1. Class-Intervals. 2. Tally Marks. 3. Frequency

Only class-Intervals will be mentioned. Other columns will be empty.

Class Interval	Tally Marks	Frequency
45 - 51		
52 - 58		
59 - 65		
66 - 72		

Step 4: Find the frequency using tally method.

Tally Marking: We tick the members one by one. For each tick we put a tally mark (vertical line-segment) in the Tally Marks column against the corresponding class-interval to which the ticked members belong as explained in solving example 1, given an page 235.

Class Interval	Tally Marks	Frequency
45 - 51	𝍸 ||||	
52 - 58	||||	
59 - 65	𝍸 ||	
66 - 72	𝍸 ||	

Example 1. The following are the marks (out of 100) obtained by 27 students of a school in Mathematics Test.

67, 45, 68, 56, 69, 50, 65, 51, 66, 51, 53, 54, 65, 53, 48, 54, 63, 62, 61, 59, 67, 45, 68, 50, 69, 50, 65. Make a frequency distribution table taking 4 class-intervals of equal size.

Solution: Size of class interval $= \frac{69 - 45}{4} = \frac{23}{4} = 6$ (approximately)

Thus, 4 class interval arc: 45 – 51, 52 – 58, 59 – 65 and 66 – 72.

Then, we use tally marking method for finding frequency as follows:

First item (member) of the raw-data is '67', we put a tick mark (67) on it and put a tally mark (|) in the column of 'Tally Mark' against class-interval 66 - 72 as shown above because '67' falls in this interval.

Then we have to tick second item (45) in the raw data and put a tally mark against the class-interval 45-51.

Next we tick third member (68) of the raw data and put a tally mark against the class-interval 66-72 as shown above in the table.

In this way, we go on repeating the process until we reach the last item of raw data. Upto four tallies (||||) in a class-interval, we draw vertical line segments, but for the fifth tally-mark in the same row we draw a diagonal line-segment cutting the previous four tallies as 𝍸.

This grouping of tallies in five makes the counting of tallies simple. In the end, we complete the column of 'Frequency' by counting the tally marks in each class-interval. The complete final table is given below:

Class Interval	Tally Marks	Frequency
45 - 51	𝍸 ||||	9
52 - 58	||||	4
59 - 65	𝍸 ||	7
66 - 72	𝍸 ||	7
	Total	27

12.1.3 Construct a Histogram Representing Frequency Table.

A Histogram is constructed from a frequency table.

It consists of a set of adjacent rectangles for each class. All the rectangles are of equal width but their heights are different, as they represent frequencies of the classes.

To construct a histogram from a grouped data the following steps are followed:

(i) Draw X-axis and Y-axis.

(ii) Mark class boundaries of the classes along X-axis.

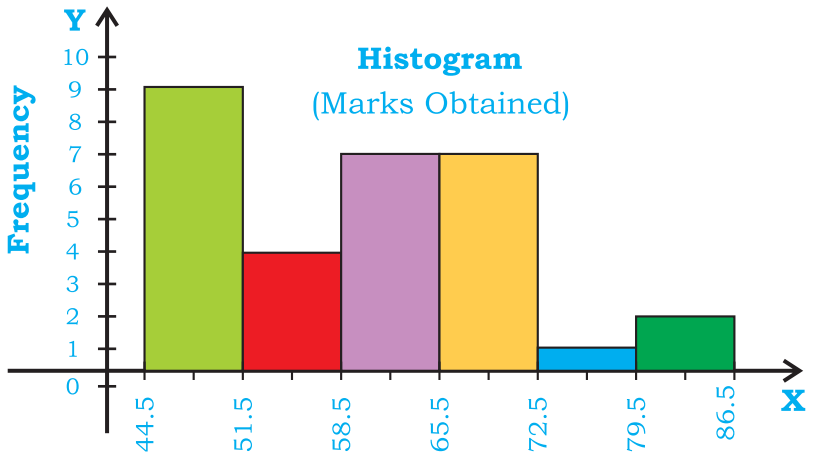
(iii) Mark frequencies along Y-axis.

(iv) Draw a rectangular bar for each class so that the height of the bar drawn for each class is equal to the frequency of the class.

For example, the data given below is shown in frequency table.

67, 45, 68, 56, 69, 50, 65, 51, 66, 51, 53, 54, 65, 53, 48, 54, 63, 62, 61, 59, 75, 45, 68, 50, 69, 50 65, 76, 83, 83

Class Boundaries	f	Mid Point x
44.5 - 51.5	9	48
51.5 - 58.5	4	55
58.5 - 65.5	7	62
65.5 - 72.5	7	69
72.5 - 79.5	1	76
79.5 - 86.5	2	83
Total	**30**	--



Example 2. 26 Students of class VIII in a school scored the following marks out of 20 marks, in a test in Mathematics.

4	10	12	16	2	8	14	16	18	20	13	4	12
6	8	10	14	16	15	16	16	14	8	12	10	11

Suppose the students are graded as:

Marks 16-20, Grade-A Marks 11-15, Grade-B

Marks 06-10, Grade-C Marks 01-05, Grade-D

Draw a Histogram and answer the following questions.

(a) How many students got Grade-A?

(b) How many students got Grade-B?

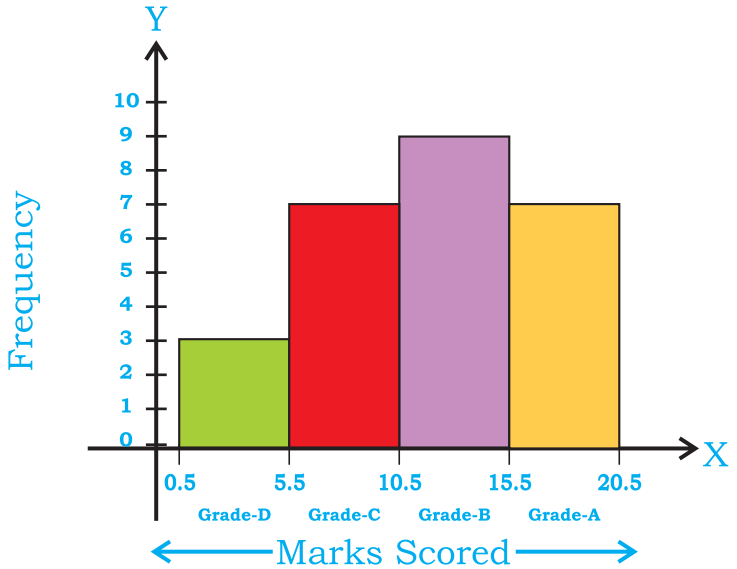
(c) Which grade was obtained by maximum number of students?

Solution: Here the data is not grouped. Thus we have to construct the frequency Table and find class boundaries (Class-Intervals are already given to us). The table is as under:

FREQUENCY DISTRIBUTION TABLE

Class Interval	Tally Marks	Frequency	Class Boundaries
1 - 5	|||	3	0.5 - 5.5
6 - 10	𝍸 ||	7	5.5 - 10.5
11 - 15	𝍸 ||||	9	10.5 - 15.5
16 - 20	𝍸 ||	7	15.5 - 20.5

We construct the rectangles following the steps and get histogram.



Heights of the rectangles indicate that: (a) 7 students got Grade A (b) 9 students got Grade B (c) Maximum number of students got Grade B.

Note: A histogram looks similar to a bar-graph but there is some difference between them.

The bars in a bar-graph are disjoint with one another and their heights represent values of individual items but in a histogram, rectangular bars are not disjoint but adjacent to one another and their heights represent frequencies of the classes.

EXERCISE 12.1

1. Consider the weights in Kg of 30 boy scouts selected from different schools: 39, 40, 36, 32, 38, 40, 42, 35, 41, 29, 40, 42, 43, 39, 37, 35, 33, 31, 32, 34, 36, 42, 45, 43, 45, 47, 42, 44, 46, 40.

Make a frequency distribution table taking 5 class intervals of equal size by tally marks. Construct its Histogram.

2. The number of units of electricity consumed by 25 households of a colony are listed below. Construct a frequency table with 5 classes of equal size and then draw a histogram.

700, 720, 750, 740, 750, 730, 690, 695, 690, 695, 700, 710, 705, 690, 685, 695, 690, 705, 710, 700, 680, 677, 720, 725, 730.

3. Given below are the marks secured by 28 students of class VIII in a monthly test in Mathematics:

60, 58, 62, 61, 59, 65, 67, 67, 64, 60, 63, 70, 67, 69, 70, 80, 79, 78, 71, 82, 50, 45, 55, 56, 52, 54, 59, 60.

Prepare its Frequency Table with 5 classes of equal size and then construct a histogram.

4. Prepare a frequency Table with 4 classes of equal size using tally method from the data of rain-fall (in cm) for the last 25 years.

15, 25, 16, 18, 9, 5, 7, 14, 21, 23, 4, 11, 22, 3, 5, 5, 12, 25, 13, 17, 5, 26, 7, 9, 24. Construct its Histogram.

5. The following frequency table represents the scores obtained by a group of commerce students in the computer laboratory.

Marks	26-30	31-35	36-40	41-45	46-50	51-55
No. of Students	05	17	13	24	15	14

Draw a Histogram for the above distribution.

6. Construct a Histogram for the distribution of daily earnings in rupees of workers given below.

Earnings	501-600	601-700	701-800	801-900	901-1000	1001-1100
No. of Workers	10	15	20	25	30	35

12.2 Measures of Central Tendency

We know that the information can easily be understood, when the raw data is converted into a frequency distribution. The information given in the data can be further condensed to a single representative value for the entire distribution. As this value lies in the central part of a data, so it is called measure of central tendency.

12.2.1 Describe Measures of Central Tendency

The most commonly used measures of central tendency are :
(i) Mean (ii) Weighted Mean (iii) Median (iv) Mode

12.2.2 Calculate Mean, Weighted Mean, Median and Mode for ungrouped data

(A) Mean (Average): Mean (Average) is said to be the most popular and an ideal average. For example, the mean temperature of a city remained 39 °C in the month of May means that the temperature will be above 39 °C for some days and it will be below 39 °C for other days of May.

Similarly, the mean monthly income of a worker is Rs 30,000. It means that his mean income per day is Rs 1,000. Actually his income remains more than Rs 1,000 for some day and less than Rs 1,000 in some other days of the month.

The method to find the mean is to divide the sum of all the measures of ungrouped data by the total number of all the measures. The formula for mean is as under.

$$\text{Mean} = \frac{\text{Sum of all the measures}}{\text{Number of measures}}$$

or $\overline{X} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$ (x_i stands for in the measures)

or $X = \frac{\text{Sum of the values of all observations}}{\text{Number of observations}}$

Note: Mean is denoted by $\overline{X}$ and sum by $\sum$. Hence $\overline{X} = \frac{\sum_{i=1}^{n} x_i}{n}$

Observations:

(i) Mean for 1, 3, 5, 7, 9 is **5**
(ii) Mean for 6, 8, 10, 12, 14 is **10** (adding 5 to each element)
(iii) Mean for 0, 2, 4, 6, 8 is **4** (subtracting 1 from each element)
(iv) Mean for 5, 15, 25, 35, 45 is **25** (why?)
(v) Mean for 1,3,4,5,6,7,9 is also **5.**
(vi) Mean for 2,4,5,6,8 is also **5** (why?)

Note: For a given data mean is unique but the same number may be mean of many different data sets (i), (v), (vi)

Example 1. A student obtained 84, 90, 72, 60, 74, 50, 40 and 52 marks in 8 different subjects in annual examination of class VIII. Find his mean marks.

Solution: Mean = $\frac{\text{Sum of marks in all subjects}}{\text{Number of Subjects}} = \frac{\sum x_i}{n}$

Or $\overline{X} = \frac{84+90+72+60+74+50+40+52}{8} = \frac{512}{8} = 64$ marks.

Note 1. Mean 64 is not present among the obtained marks. In fact mean may or may not be present in the data.

Note 2. $\overline{X}$, Σx and n are three terms in the above formula. If any two of these are given, we can find the third unknown term.

Example 2 (i). If Mean = 6 and Σx = 18, then find n.

Solution: Since $\overline{X} = \frac{\sum x}{n} \Rightarrow n = \frac{\sum x}{\overline{X}} = \frac{18}{6} = 3$

Example 2 (ii). If $\overline{X}$ = 4 and n = 5, find Σx.

Solution : Since $\overline{X} = \frac{\sum x}{n} \Rightarrow \Sigma x = n\ \overline{X} = 5\,(4) = 20.$

Example 2 (iii). If $\overline{X}$ = 6, n = 4 and three terms in the data are 2, 8, 10, then find the fourth missing term.

Solution: We are given $\overline{X}$ = 6, n=4 and $\Sigma x = 2 + 8 + 10 + x_4 = 20 + x_4$

We know that: $\overline{X} = \frac{\sum x}{n} \Rightarrow \Sigma x = n\ \overline{X} \Rightarrow 20 + x_4 = 4\,(6) = 24$

$\Rightarrow x_4 = 24 - 20 = 4.$ Hence the fourth term is 4.

Example 2 (iv). The mean age of the students of class VIII is 13 years. If the age of Maths teacher is also added, the mean age rises to 14 years. If number of students in the class is 30, find the age of the teacher.

Solution: Mean age of students = $\overline{X}_s$ = 13, n_s = 30 .

$\therefore$ Sum of ages of students = $\Sigma x = n_s \times \overline{X}_s = 30 \times (13) = 390$ years.

Now Sum of ages of Students + Age of Teacher

$= n_{s+t} \times \overline{X}_{s+t} = 31 \times 14 = 434$ years.

Where n_s denotes number of students and $\overline{X}_{s+t}$ denotes mean age of students and teacher together. $\therefore$ Age of Teacher = 434 – 390 = 44 years.

(b) Weighted Mean

When all the values of the given data have same importance, then we use the mean. However, when different values have different importance then these values are known as weights.

For example (i) If Mathematics has more importance than History and English has more importance than Mathematics then we say $W_{eng} > W_{maths} > W_{History}$.

(ii) Vote of chairman has more importance than vote of a member.

Note: Weighted mean changes if weights are changed.

Let $x_1, x_2, x_3, x_4, \ldots, x_n$ have weights $w_1, w_2, w_3, \ldots, w_n$, then we can find weighted mean as follows:

$$\text{Weighted Mean} = \frac{x_1.w_1+x_2.w_2+x_3.w_3+\ldots+x_n.w_n}{w_1+w_2+w_3+w_4+\ldots.+w_n} \Rightarrow \bar{X}_w = \frac{\Sigma x_i\, w_i}{\Sigma w_i}$$

Example 3. Students of class VIII purchased books of different subjects as under:

Title of the book	Maths	Science	English	Urdu	Social Studies
Price of the book in rupees	72	80	60	40	50
Quantity of the book	5	4	6	3	2

Here quantity shows importance.

Find weighted Mean price of the book.

(Hint: Calculate Weighted Mean).

Solution: Weighted Mean = $\bar{X}_w = \frac{\Sigma x_i\, w_i}{\Sigma w_i}$

$$= \frac{(72 \times 5) + (80 \times 4) + (60 \times 6) + (40 \times 3) + (50 \times 2)}{5 + 4 + 6 + 3 + 2}$$

$$= \frac{360 + 320 + 360 + 120 + 100}{20} = \frac{1260}{20} = 63 \text{ rupees}$$

Thus, weighted Mean is 63, i.e., Mean price of the books is Rs 63.

(c) Median

Median is the middle value of the data. It divides the data into two parts, i.e., first half of the data is before the median value and the second half is after the median value. For finding the median value of a given ungrouped data first we have to arrange the given data in order.

(i) If the number of items (n) in a given data is odd, then the middle term will be the median, i.e., $(\frac{n+1}{2})^{th}$ term.

(ii) If the number of items in a given data is even, then mean of the two middle terms will be the median.

Example 4. Find the median of the data: 2, 10, 6, 8, 9, 5, 3, 7, 4.

Solution: First of all, given data is arranged in ascending order.

2, 3 , 4, 5, $\boxed{6}$, 7, 8, 9, 10.

Here the given data consists of 9 terms (9 is an odd number)

Therefore Median = $\left(\frac{n+1}{2}\right)$ th value. Here $n = 9$.

Hence Median = $\left(\frac{9+1}{2}\right)$ th value = $\left(\frac{10}{2}\right)$ th value = 5th value = 6

Example 5. Find the median of 7, 10, 11, 13, 4, 8, 14, 12, 6, 3.

Solution: First of all the given data is arranged in descending order:

14, 13, 12, 11, $\boxed{10, 8}$, 7, 6, 4, 3.

In the given data there are 10 terms and it is an even number. As fifth and sixth terms are the two middle terms. So, mean of these terms (10 and 8) is the median, i.e.,

$$\text{Median} = \frac{10+8}{2} = \frac{18}{2} = 9.$$

Thus, the median of the given data is 9.

[**Note:** Result will be the same if the data is arranged in ascending order]

Observations:

(i) Median for 3, 5, $\boxed{7}$, 10, 15, is **7**

(ii) Median for 13, 15, 17, 20, 25 is **17** (adding 10)

(iii) Median for 1, 3, 5, 8, 13 is **5** (why ?)

(iv) Median for 30, 50, 70, 100, 150 is **70** (why?)

(v) Median for 2,4, $\boxed{7}$,10,15 is also **7** .

(vi) Median for 1,3,5, $\boxed{7}$,10,15,30 is also **7** .

Note: For a given data median is unique; but the same number may be median of many different data as mentioned in above observations (sets (i), (v), (vi)).

(D) Mode

Mode is that value which occurs maximum (most) number of times in a group of data. It is that value which has the greatest frequency (most common value) in a given grouped data. When none of the value is repeated, there is no mode in the data. When there are two or more values of the same greatest frequency, there are two or more modes in the data.

Example 6. The marks obtained by a group of 12 students are given as: 70, 50, 40, 65, 45, 65, 70, 65, 50, 45,70, 65. Find the mode of the data.

Solution: Given data is arranged in ascending order.

$$40,\ \underbrace{45, 45}_{2\text{ times}},\ \underbrace{50, 50}_{2\text{ times}},\ \underbrace{65, 65, 65, 65}_{4\text{ times}},\ \underbrace{70, 70, 70}_{3\text{ times}}$$

In the given data 65 comes four times, whereas other numbers occur less number of times.

Hence, 65 marks is the mode of the marks obtained by students.

Example 7. The score of eleven players is 13, 18, 12, 17, 13, 12, 16, 12, 13, 13,12. Find the mode of the data.

Solution: Arrange the given score of 11 players in descending order.

$$18, 17, 16,\ \underbrace{13, 13, 13, 13}_{4\text{ times}},\ \underbrace{12, 12, 12, 12}_{4\text{ times}}$$

It is observed that the scores 12 and 13 are repeated four times each. Hence, there are two modes 12 and 13 in the given data.

Note1: Mean and median are always unique for a given data.

Note2: There may be more than one mode. Sometimes there may not be any mode at all.

For example: B = {1,2,5} has **no** mode.

C = {1,2,2,2,5,5,5,8,8} has **2** modes. i.e., 2 and 5.

D = {Good, Poor, Fair, Fair, Dull} has **1** mode. i.e., Fair.

Note: Mode may exist for qualitative data.

Mean and median does not exist for qualitative data.

EXERCISE 12.2

1. Find the mean value of the following data:

(i) 3, 8, 5, 4, 6, 0, 7, 1, 2

(ii) 11, 13, 15, 7, 9, 1, 3, 5

(iii) 11, 16, 13, 17, 10, 15, 18, 14, 12

(iv) 4, 16, 32, 8, 40, 12, 20, 28, 24, 36

(v) 1.2, 2.4, 3.6, 4.8, 5.1, 6.3, 7.5, 8.7

1. B. Find the value of required terms in the following :

(i) $\overline{X}=12, \Sigma x_i=132, n=?$ **(ii)** $n=15, \Sigma x_i=75, \overline{X}=?$

(iii) $\Sigma x_i=96, \overline{X}=8, n=?$ **(iv)** $\overline{X}=13, n=11, \Sigma x_i=?$

2. Find weighted mean of the following.

(i)

Number (x)	10	20	30	40	50	60	70
Weight (w)	1	2	1	1	2	2	1

(ii)

Number (x)	0	2	4	6	8	10	12
Weight (w)	1	2	3	4	5	3	2

(iii)

Number (x)	1	3	5	7	9	11
Weight (w)	3	2	1	1	2	1

(iv)

Number (x)	5	15	25	15	35	45
Weight (w)	2	1	2	1	2	2

3. Find median of the following:

(i) 17, 19, 11, 5, 3, 7, 9, 13, 15
(ii) 45, 5, 15, 20, 35, 40, 10, 25, 30
(iii) 22, 2, 20, 4, 18, 6, 16, 8, 14, 10, 12
(iv) 35, 21, 49, 77, 45, 27, 33, 55, 63, 81, 99
(v) 39, 78, 91, 75, 104, 105, 19, 29, 99, 79, 89
(vi) 2.5, 3.5, 1.5, 4.5, 5.5, 5.0, 9.9, 9.5, 10.5, 11.5, 6.5, 7.5, 5.5

4.A. Find mode of the following:

(i) 14, 21, 27, 14, 28, 29
(ii) 24, 31, 37, 24, 38, 39
(iii) 23, 29, 41, 23, 29, 77, 55, 29
(iv) 20, 26, 38, 20, 26, 74, 52, 26
(v) 71, 51, 81, 71, 51, 57, 71, 81, 93, 51, 71
(vi) 93, 75, 85, 89, 77, 55, 33
(vii) 930, 750, 850, 890, 770, 550, 330

4.B. Fill in the blanks:

(i) 3,3,4,4,5,8,7,8, _____, 9, if mode is 8.
(ii) 4,5,12,10,20 _____ , 5, if median is 8.
(iii) 6,9,7,10, _____, 4, if mean is 8.

5. If the sum and mean of some digits be 30 and 5, find the number of digits.

6. The mean of 4 numbers is 45. The first two numbers are 64 and 36. If the remaining two numbers are equal to x, find the value of x.

7. Find a missing number, if median is 5 for:

(i) 1, 2, 5,___,11. **(ii)** 2, ___,5, 8, 11 **(iii)** 1,2,5,____,11,20
(iv) 1,2,4,____,11,20

8. Find the missing number, if mode is 5 for:

(i) 1, 2, 5,___,11. **(ii)** 2, ___,5, 8, 11 **(iii)** 1,2,5,____,11,20
(iv) 1,2,5,5,____,11. **(v)** 1,2,5,5,____,11
(Any number not equal to 1,2,11)

12.2.3 To solve Real Life Problems involving mean, weighted mean, median and Mode.

Example 1. A businessman sent 125 crates of Sindhrri mangoes from Mirpurkhas to Karachi. Refer the table given below. Find weighted mean of rotten mangoes in each crate. (No. of crates are weights)

Number of rotten mangoes (x) per crate	0	1	2	3	4	5	6	7
Number of crates (w)	14	25	29	28	20	15	5	2

Solution:

Weighted Mean $\overline{X}_w = \dfrac{\Sigma x_i\, w_i}{\Sigma w_i}$

$$= \frac{0\times14+1\times25+2\times29+3\times28+4\times20+5\times15+6\times5+7\times2}{14+9+29+28+20+15+5+2}$$

$$= \frac{0+25+58+84+80+75+30+14}{122} = \frac{366}{122} = 3.$$

Therefore, the rotten mangoes per crate are 3.

Example 2. The weights of 20 students (in Kilograms) are given below:
60, 61, 67, 68, 59, 58, 55, 58, 59, 56, 60, 61, 66, 65, 64, 63, 60, 59, 58, 56.

Find: **(i)** The mean weight of students.
(ii) The mode or modes of weight of students.
(iii) The median weight of students.

Solution: First we have to arrange the given data in order, say in descending order:

68, 67, 66, 65, 64, 63, 62, 61, 61, 60, 60, 60, 59, 59, 59, 58, 58, 58, 56, 56
(61, 61: 2 times; 60, 60, 60: 3 times; 59, 59, 59: 3 times; 58, 58, 58: 3 times; 56, 56: 2 times)

(i) Sum=68+67+66+65+64+63+62+2(61)+3(60)+3(59)+3(58)+2(56) = 1220

$$\therefore \overline{X} = \frac{\Sigma \bar{x}}{n} = \frac{1220}{20} = 61.$$

Thus, mean weight of students is 61Kg.

(ii) In the given data, there are three modes (60kg, 59kg and 58 kg) because each of the numbers 60, 59 and 58 occur three times.

(iii) There are 20 numbers in the data.
Hence, the median is the mean of the 10th and 11th numbers in the data. i.e., Median = $\frac{60+60}{2}$ =60.

Thus median weight of students is 60Kg.

EXERCISE 12.3

1. A person purchased the following vegetable items:

S. No.	Vegetable Items	Quantity (in Kg)	Rate/Cost Per Kg (in Rupees)
(i)	Potatoes	12 Kg	Rs 25
(ii)	Onions	8 Kg	Rs 30
(iii)	Tomatoes	5 Kg	Rs 60
(iv)	Carrots	3 Kg	Rs 40
(v)	Lady Fingers	4 Kg	Rs 80

Find weighted mean cost of vegetable items per Kg.

2. The heights of 20 students (in centimetres) of Class VIII are given below:

130, 135, 140, 145, 142, 141, 140, 135, 136, 137, 132, 134, 136, 138, 140, 142, 144, 143, 139, 138.

Compute the mean, mode and median value of height of students. Is Mean ≥ Median ≥ Mode?

3. Twenty crates of Sindhrri mangoes were sent to Sukkur from Hyderabad. The number of rotten mangoes in each crate, respectively, is given below:

0, 2, 1, 0, 3, 2, 1, 0, 4, 1, 2, 3, 0, 2, 3, 4, 1, 2, 3, 3.

Find the mean, mode and median number of rotten mangoes per crate. Is Mean ≥ Median ≥ Mode?

4. Forty five students appeared in Science and Mathematics test. The number of marks obtained by them is given below .

Number of Marks obtained	10	20	30	40	50	60	70	80	90
Number of Students in Mathematics	5	4	7	8	6	3	4	5	3
Number of Students in Science	4	4	8	8	5	3	2	7	4

Find the weighted mean of the marks obtained in each subject. In which subject did the performance of students remain better? (Number of students are weights).

5. The table given below shows the size of 100 families.

Size of family	2	3	4	5	6	7	8	9	10
Number of families	24	32	20	10	5	4	2	2	1

Compute weighted mean of size of a family. (No. of families are weights).

REVIEW EXERCISE 12

1. For each statement given below, encircle the correct answer.

(i) The mean value of 7, 6, 5, 5, 0, 1 is:

(a) 3 (b) 4 (c) 5 (d) 6

(ii) The modal value of data 2, 3, 4, 3, 5, 3 is:

(a) 1 (b) 2 (c) 3 (d) 4

(iii) The median of data 10, 5, 4, 3, 2 is:

(a) 4 (b) 3 (c) 2 (d) 5

(iv) The median value of data 0, 2, 4, 8, 6, 2 is:

(a) 2 (b) 0 (c) 4 (d) 3

(v) The number of modes in data, 2, 4, 6, 6, 8, 2, 4 is:

(a) 1 (b) 2 (c) 3 (d) 4

(vi) The formula for finding mean is:

(a) $\overline{X} = \frac{\Sigma x}{n}$ (b) $\overline{X} = \frac{n}{\Sigma x}$ (c) $\overline{X} = \Sigma x.n$ (d) $n = \frac{\overline{X}}{\Sigma x}$

(vii) The value that occurs most frequently in the data is known as:

(a) Mean (b) Median

(c) Frequency Distribution (d) Mode

(viii) If $\overline{X} = 6, n = 5,$ then Σx is equal to

(a) 5 (b) 6 (c) 30 (d) $\frac{5}{6}$

(ix) A method for organizing and summarizing the primary data into grouped data is called.

(a) Frequency Distribution (b) Mean

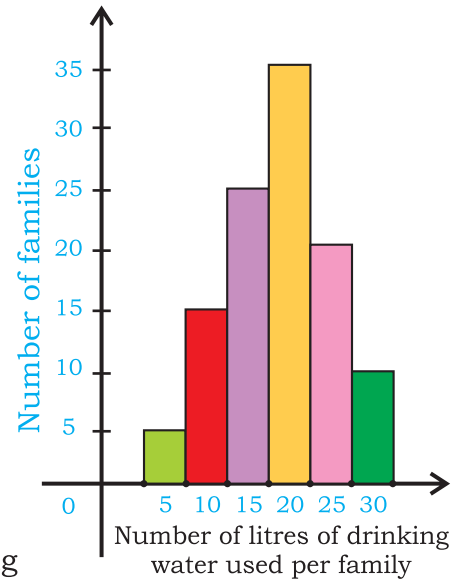
(c) Median (d) Mode

(x) Measure of central tendency for qualitative data is:

(a) Mean (b) Weighted mean

(c) Mode (d) Median

2. The diagram illustrates the number of litres of drinking water per family of 110 families in a certain plaza.



(i) How many numbers of families used 25 to 30 numbers of litres of drinking water per family?

(ii) How many numbers of families used 5 to 10 numbers of litres of drinking water per family?

(iii) How many numbers of families used 15 to 20 numbers of litres of drinking water per family?

(iv) How many litres of drinking water per family are used by maximum number of families?

(v) How many litres of drinking water per family are used by minimum number of families?

3. Thirty students of a class appeared in an examination. Out of 40 maximum marks, they obtained the following marks:

18	24	10	3	14	26	22	18	25	11	29	17	16	11	25
11	14	15	4	22	13	23	29	10	27	18	10	25	13	14

(i) Form a frequency distribution table from the given information, with 5 classes of equal size.

(ii) Find the model class and size of each class.

(iii) Draw a histogram of the result of the examination.

4. Prepare the frequency distribution table with 4 equal class intervals, using tally method as well as listing method from the data of marks obtained by 27 students:

60, 62, 66, 67, 68, 66, 60, 73, 67, 69, 70, 63, 64, 63, 60, 50, 55, 59, 60, 65, 70, 72, 69, 59, 50, 60, 70.

Draw its histogram.

SUMMARY

- Information handling is a part of Statistics.
- Statistics is the branch of study concerned with the collection and classification of numerical facts.
- While collecting the data, if some quantities are repeated, the number of repetition is called frequency of that quantity.
- A method of organizing and summarizing the primary data into grouped data is called Frequency Distribution.
- A histogram is a visual representation of a frequency distribution.
- A histogram is a vertical bar graph in which there is no gap between the bars in rectangular form.
- Central tendency is the measure that identifies a single value as representative of an entire data.
- In general four forms of central tendency are used i.e. Mean, weighted mean, mode and median.
- Formula for finding the mean is: $\overline{X} = \frac{\Sigma x}{n}$, i.e., Sum of all the values in the data divided by the number of values in the data..
- Mode is that value which occurs maximum number of times in a set of data. It may or may not exist.
- Median is that value which divides the data in two equal parts (Unique).
- We arrange the data in ascending or descending order to find the median.

  (i) If the number of items in the data are odd, the middle term will be median.

  (ii) If the number of items are even, then the mean of two middle terms will be the median.

LIST OF SYMBOLS AND ABBREVIATIONS

$\varnothing$		Null set, Empty set or Void set
$\mathbb{N}$	$= \{1, 2, 3, \ldots.\}$	Set of Natural Numbers
$\mathbb{W}$	$= \{0, 1, 2, 3, \ldots\}$	Set of Whole Numbers
$\mathbb{Z}$	$= \{0, \pm1, \pm2, \ldots\}$	Set of Integers
$\mathbb{Q}$	$= \{x \mid x = \frac{p}{q},\ pq \in \mathbb{Z}, q \neq 0\}$	Set of Rational Numbers
$\mathbb{E}$	$= \{0, \pm2, \pm4, \pm6, \ldots\}$	Set of Even integers
$\mathbb{O}$	$= \{\pm1, \pm3, \pm5, \ldots\}$	Set of Odd integers
$\mathbb{P}$	$= \{2, 3, 5, 7, \ldots\}$	Set of Prime Numbers
$\subseteq$	=	Sign of subset
$\subset$	=	Sign of proper subset
P (A)	=	Power set of a set A; Set of all the possible subsets of set A.
$\cap$	=	Intersection
$\cup$	=	Union
$\wedge$	=	And
$\mathbb{U}$	=	Universal Set
$A' = \mathbb{U} - A$	=	Complement of Set A
$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$	=	Set of Real Numbers
$\mathbb{Q}' = \mathbb{R} - \mathbb{Q}$	=	Set of Irrational Numbers
$\sqrt{\ }$	=	Square Root sign
$\sqrt[3]{\ }$	=	Cube Root sign
0 and 1	=	Binary Digits
0,1,2,3 and 4	=	Numerals of base 5 system
0,1,2,3,4,5,6 and 7	=	Numerals of base 8 or octal system
0,1,2,3,4,5,6,7,8 and	9 =	Numerals of base 10, Decimal System
ATM	=	Automated Teller Machine
$	=	US Dollars
£	=	Pound Stierling
﷼	=	Saudi Riyal
Rs	=	Rupees (Pakistani)
€	=	Euro
¥	=	Yuan (Chinese)
¥	=	Yen (Japanees)
I = PRT	=	Profit = Principal × Rate × Time
C.P	=	Cost Price
S.P	=	Selling Price
M.P	=	Marked Price
$\overline{AB}$	=	Line Segment AB

LIST OF SYMBOLS AND ABBREVIATIONS

$\overleftrightarrow{AB}$	=	Line AB
$\perp$	=	Perpendicular
$\parallel$	=	Parallel
m $\overline{AB}$	=	Measure (Length) of line segment AB
m$\angle$A	=	Measure of Angle A
$\parallel^m$	=	Parallelogram
a, b, c; $\overline{BC}$, $\overline{AC}$, $\overline{AB}$		Sides of a $\triangle$ABC
$\triangle$	=	Triangle
$\blacktriangle$	=	Area of Triangle
S	=	Semi-perimeter of a Triangle
s	=	Height of slant side of a cone
r	=	Raduis of a circle or sphere
h	=	Height of a cylinder
V	=	Volume
π	Pi =	$\frac{22}{7}$ (approximately)
Q.E.D	=	Quod Erot Demonstrandum Meaning 'which was to be proved.'
$\leftrightarrow$	=	One-One (or 1-1) correspondence
$\cong$	=	Congruence
S.A.S	=	Side Angle Side
Sin θ	=	Sine of the Angle θ (Theta)
Cos θ	=	Cosine of the Angle θ
Tan θ	=	Tangent of the Angle θ
Cot θ	=	Cotangent of the Angle θ
Sec θ	=	Secant of the Angle θ
Cosec θ or Csc θ =		Cosecant of the Angle θ
f	=	Frequency
$\overline{X}$	=	Mean
Σx_i	=	Sum of Measures
n	=	Number of Measures
$\overline{X}_\omega$	=	Weighted Mean
𝍸	=	Tally Marks , 5

Revision Exercise I

1. (i) A={1,2,3,4,5} (ii) B={ – 1,0,1,2}
(iii) C = {11,13,17,19,23,29}

2. (i) A, C (ii) B,D,E
(iii) A and D, A and C, A and E, B and E, C and E, B and D
(iv) A and B, B and C, D and E.

Exercise 1.1

1. (i) $A \subseteq B$ (ii) $C \subseteq D$ (iii) $S \subseteq T$

2. (i) $\{a\}, \{a, e\}, \{a, e, i\}$ (ii) $\{\ \}, \{x\}, \{y\}$

3. (i) Proper subsets: {2}, {4},
Improper subset: {2,4}
(ii) Proper subsets: {2}, {3};
Improper subset: {2,3}

4. (i) { }, {1}, {2}, {3}, {1,2}, {1,3}, {2,3} {1,2,3}
(ii) { }, {2} (iii) { }, {3}, {5}, {3,5}
(iv) { }, {–1}, {0}, {1}, {–1,0}, {–1,1}, {0,1}, {–1,0,1}
(v) { }, {2}, {4}, {0}, {2,4}, {2,6}, {4,6}, {2,4,6}

5. (i) P (A) = {∅,{1}, {3}, {5}, {1,3}, {1,5}, {3,5}, {1,3,5}}
(ii) P (B) = {∅, {a}, {b}, {c}, {d}, {a,b}, {a,c}, {a, d}, {b, c}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, {a,b,c,d}} (iii) P (C) = { ∅ }

6. Any set having only one member in it, is a set which was only one proper subset.

7. Null set or Empty set.

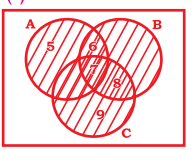
8. (i) {1,3,5,7} (ii) {2,3,5,7,11}

9. (i) n(A)=7
(ii) n(B)=2
(iii) n(C)=2 (iv) n(D)=2

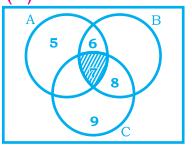
10. (i) $\mathbb{N} \subseteq \mathbb{W}$ (ii) $\mathbb{N} \subseteq \mathbb{Z}$ (iii) $\mathbb{P} \subseteq \mathbb{N}$
(iv) $\mathbb{E} \subseteq \mathbb{Z}$ (v) $\mathbb{Q} \subseteq \mathbb{Q}$ (vi) $\mathbb{E} \subseteq \mathbb{Q}$

Exercise 1.3

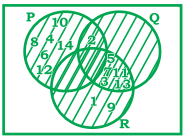
1. (i)



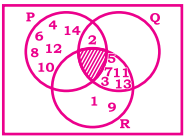
(ii)



2. (i)



(ii)



3. and **4.** Students may draw venn diagram themselves.

Review Exercise 1

1. (i) (b) (ii) (c) (iii) (c) (iv) (b)

2. P (A) = {∅, {x}, {y}, {z}, {x,y}, {x,z}, {y, z}, {x,y,z}}

5. (i) (b) (ii) (d) (iii) (c)

6. (iv), {x, y}; (vi) ∅

Exercise 2.1

A. Rational Numbers: 1,2,3,4,6,8,9
Irrational Numbers: 5,7,9,10,12

B. Terminating Numbers: (iii), (vi), (viii), (ix), (xi), (xii)
Non-Terminating Numbers: (i), (ii), (iv), (v), (vii), (x)

Exercise 2.2

A. **1.** 25 **2.** 169. **3.** 361 **4.** 1521 **5.** 2025 **6.** 3354 **7.** 3969 **8.** 6241 **9.** 9409 **10.** 11664

B. **1.** 16, 25 **2.** 25, 36, 49 **3.** 49, 64 **4.** 81

C. **1.** $7^2 = 1+2+...+7+6...+1=49$
2. $9^2 = 1+2+...+9+8...+1=81$

3. $15^2 = 1+2+...+15+14...+1=225$
4. $11^2 = 1+2+...+11+10+...+1=121$
5. $13^2 = 1+2+...+13+12+...+1=169$
6. $8^2 = 1+2+...7+8+7+...+1=64$
7. $12^2 = 1+2+...+12+11...+1=144$
8. $16^2 = 1+2+...+16+15...+1=256$
9. $20^2 = 1+2+...+20+19...+1=400$
10. $27^2 = 1+2+...+27+26...+1=729$

Exercise 2.3

A. **1.** 31 **2.** 36 **3.** 587 **4.** 904
5. $\frac{9}{11}$ **6.** $\frac{11}{14}$ **7.** $\frac{21}{44}$ **8.** $1\frac{4}{121}$
9. 15.8 **10.** 0.64 **11.** 54.6 **12.** 11.45

B. **1.** 32 **2.** 119 **3.** 998 **4.** 3214
5. $\frac{13}{17}$ **6.** $\frac{35}{53}$ **7.** $\frac{171}{236}$ **8.** $1\frac{1}{4}$
9. 25.47 **10.** 13.45 **11.** 26.98
12. 87.256

Exercise 2.4

A. **1.** 1.41 **2.** 2.23 **3.** 3.60
4. 11.87 **5.** 13.41 **6.** 1.58
7. 12.36 **8.** 0.11 **9.** 4.80
10. 11.18

B. **1.** 2.449 **2.** 2.828 **3.** 3.316
4. 12.449 **5.** 14.317 **6.** 2.701
7. 11.215 **8.** 38.326 **9.** 359.232
10. 57.924

Exercise 2.5

1. 2 **2.** 3 **3.** 3 **4.** 3
5. 3 **6.** 4 **7.** 4 **8.** 5
9. 4 **10.** 4 **11.** 5 **12.** 5

Exercise 2.6

1. Rs 92 **2.** 104 m **3.** No. 40
4. 21.5 metres **5.** 2500 metres

Exercise 2.7

A. **1.** 1,000 **2.** 8,000 **3.** 2,7000
4. 64,000 **5.** 12,5000 **6.** 216,000
7. 512 **8.** 1,728 **9.** 3,375
10. 5,832 **11.** 2,304 **12.** 15,625

B. **1.** $\frac{1}{8}$ **2.** $\frac{8}{9}$ **3.** $\frac{27}{64}$
4. $\frac{64}{125}$ **5.** $2\frac{10}{27}$ **6.** $15\frac{5}{8}$
7. $1\frac{61}{64}$ **8.** $10\frac{37}{216}$ **9.** $1\frac{331}{1000}$
10. $9\frac{7507}{8000}$ **11.** $1\frac{271}{729}$ **12.** $37\frac{1}{27}$

C. **1.** 0.064 **2.** 0.729 **3.** 1.729
4. 4.096 **5.** 9.261 **6.** 15.625
7. 0.000001 **8.** 0.000008
9. 0.000125 **10.** 1.404928
11. 1.157625 **12.** 1.030301

D. **1.** yes **2.** yes **3.** yes **4.** yes
5. yes **6.** yes **7.** yes **8.** yes
9. no **10.** yes **11.** yes **12.** yes

Exercise 2.8

A. **1.** 4 **2.** 7 **3.** 11 **4.** 8
5. 9 **6.** 42 **7.** 15 **8.** 24
9. 25 **10.** 33 **11.** $\frac{1}{2}$ **12.** $1\frac{1}{3}$
13. $1\frac{1}{4}$ **14.** $1\frac{1}{5}$ **15.** $\frac{7}{9}$

Review Exercise 2

1. (i) (c) (ii) (c) (iii) (a) (iv) (b)
(v) (c) (vi) (a) (vii) (a) (viii) (c)
(ix) (a) (x) (c) (xi) (a) (xii) (c)

2. (i) 3 (ii) 3 (iii) 4

3. (i) $4\frac{3}{17}$ (ii) $5\frac{1}{3}$ (iii) $10\frac{1}{13}$

(iv) 0.231 (v) 0.452 (vi) 8.1

4. 402 m

5. (i) 15 (ii) 12 (iii) $\frac{7}{8}$ (iv) $1\frac{2}{9}$

Exercise 3.1

1. **i.** A system of numbers in which only two digits 0 and 1 are used.
$0_2, 1_2, 10_2, 11_2, 100_2, 101_2, 110_2, 111_2$

ii. A system of numbers in which only five digits. 0,1,2,3 and 4 are used.
$0_5, 1_5, 2_5, 3_5, 4_5, 10_5, 11_5, 12_5, 13_5, 14_5, 20_5, 21_5$

iii. A system of numbers in which only eight digits 0,1, 2, 3, 4, 5, 6 and 7 are used.
$0_8, 1_8, 2_8, 3_8, 4_8, 5_8, 6_8, 7_8, 10_8, 11_8, 12_8, 13_8, 14_8, 15_8, 16_8, 17_8$

iv. A system of numbers in which ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are used.

Example: $34_{10}, 57_{10}, 63_{10}$

2. **i.** $1\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0$

ii. $1\times2^3 + 1\times2^2 + 1\times2^1 + 1\times2^0$

iii. $1\times2^4 + 0\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0$

iv. $1\times2^4 + 1\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0$

v. $1\times2^5 + 0\times2^4 + 0\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0$

vi. $1\times2^5 + 0\times2^4 + 0\times2^3 + 0\times2^2 + 1\times2$
1×2^0

vii. $1\times2^5 + 1\times2^4 + 1\times2^3 + 1\times2^2 + 1\times2$
1×2^0

3. **i.** $2\times5^1 + 3\times5^0$ **ii.** $4\times5^1 + 4\times5^0$

iii. $1\times5^2 + 2\times5^1 + 4\times5^0$

iv. $1\times5^2 + 4\times5 + 2\times5^0$

v. $2\times5^2 + 1\times5^1 + 4\times5^0$

vi. $3\times5^2 + 0\times5 + 3\times5^0$

vii. $3\times5^2 + 4\times5 + 4\times5^0$

viii. $4\times5^2 + 0\times5 + 4\times5^0$

4. **i.** $3\times8^1 + 6\times8^0$ **ii.** $7\times8^1 + 1\times8^0$

iii. $1\times8^2 + 1\times8^1 + 6\times8^0$

iv. $1\times8^2 + 4\times8 + 2\times8^0$

v. $1\times8^2 + 6\times8 + 2\times8^0$

vi. $3\times8^2 + 4\times8 + 1\times8^0$

vii. $5\times8^2 + 2\times8 + 4\times8^0$

5. **i.** $2\times10^2 + 0\times10^1 + 1\times10^0$

ii. $3\times10^2 + 1\times10^1 + 9\times10^0$

iii. $4\times10^3 + 0\times10^2 + 7\times10^1 + 5\times10^0$

iv. $5\times10^4+6\times10^3+9\times10^2+7\times10^1+0\times10^0$

v. $9\times10^5 + 8\times10^4 + 0\times10^3 + 7\times10^2$
$+ 6\times10^1 + 2\times10^0$

Exercise 3.2

A. **1.** 100100_2 **2.** 1001101_2
3. 1011001_2 **4.** 10011100_2
5. 100011000_2 **6.** 111101001_2
7. 1010001110_2 **8.** 1100010010_2
9. 1111100111_2 **10.** 11001000000_2
11. 11010101001_2
12. 11100010000_2 **14.** 11111010001_2
13. 10001001000_2 **15.** 11111100100_2

B. **1.** 322_5 **2.** 343_5 **3.** 1134_5
4. 1310_5 **5.** 2440_5 **6.** 3234_5
7. 10131_5 8. 13242_5 **9.** 31012_5
10. 101313_5 **11.** 130010_5
12. 130200_5 **13.** 130031_5
14. 134402_5 **15.** 134010_5

C. **1.** 770_8 **2.** 3345_8 **3.** 47175_8
4. 101575_8 **5.** 136350_8
6. 142657_8 **7.** 167536_8
8. 232416_8 **9.** 235755_8
10. 301433_8 **11.** 257631_8
12. 301301_8 **13.** 300715_8
14. 277421_8 **15.** 303237_8

Exercise 3.3

A. **1.** 13 **2.** 27 **3.** 23 **4.** 53

5. 43 **6.** 119 **7.** 102 **8.** 103
9. 214 **10.** 253 **11.** 146 **12.** 197

B. **1.** 11 **2.** 15 **3.** 20 **4.** 7
5. 12 **6.** 41 **7.** 122 **8.** 67
9. 101 **10.** 115 **11.** 205 **12.** 611
13. 337 **14.** 508 **15.** 579

C. **1.** 39 **2.** 14 **3.** 32 **4.** 48
5. 56 **6.** 84 **7.** 162 **8.** 196
9. 296 **10.** 448 **11.** 1542 **12.** 556
13. 3456 **14.** 2309 **15.** 2595

Exercise 3.4

A. **1.** 1001_2 **2.** 1111_2 **3.** 10101_2
4. 110100_2 **5.** 11011_2 **6.** 1010_2
7. 11100_2 **8.** 11001_2 **9.** 1010110_2
10. 110000_2 **11.** 110000_2 **12.** 1001011_2

B. **1.** 1010_2 **2.** 10_2 **3.** 1000_2
4. 10101_2 **5.** 101010_2 **6.** 1110_2
7. 111_2 **8.** 111_2 **9.** 10001_2
10. 10101_2

C. **1.** 1101_2 **2.** 10100_2 **3.** 1010_2
4. 0_2

D. **1.** 1111_2 **2.** 11110_2
3. 111100_2 **4.** 1001110_2
5. 100011_2 **6.** 11010010_2
7. 10010110_2 **8.** 101101_2
9. 111100_2 **10.** 1100000001_2
11. 11010010_2 **12.** 100010100_2

Exercise 3.5

A. **1.** 1121_5 **2.** 10203_5
3. 10231_5 **4.** 112_5
5. 113434_5 **6.** 241_5
7. 10001_5 **8.** 4402_5
9. 10341_5 **10.** 22333_5
11. 30230_5 **12.** 100122_5

B. **1.** 330_5 **2.** 133_5
3. 1311_5 **4.** 1014_5
5. 3032_5 **6.** 1422_5
7. 334_5 **8.** 13123_5

C. **1.** 1000_5 **2.** 10111_5
3. 2000_5 **4.** 43124_5
5. 20324_5 **6.** 3144_5
7. 2334_5 **8.** 111_5

D. **1.** 302333_5 **2.** 242_5
3. 320042_5 **4.** 3204003_5
5. 203124_5 **6.** 220034_5
7. 114012_5 **8.** 2224222_5
9. 2312120_5 **10.** 431440_5
11. 10340220_5 **12.** 2243214_5

Exercise 3.6

1. (i) 445_8 (ii) 13755_8
(iii) 73647_8 (iv) 156344_8
(v) 726447_8

2. (i) 25_8 (ii) 5_8
(iii) 352_8 (iv) 7673_8
(v) 63744_8

3. (i) 3763_8 (ii) 66075_8
(iii) 10323050_8 (iv) 11771530_8
(v) 377645061_8

Exercise 3.7

1. (i) 111101_2 (ii) 10010010_2
(iii) 10000101_2 (iv) 1001000_2
(v) 1100111_2

2. (i) 33_5 (ii) 131_5
(iii) 34_5 (iv) 241_5
(v) 11200_5

3. (i) 202103_5 and 1100110000000_2
(ii) 141244_5 and 1011011000000_2
(iii) 214314200_5 and 11100011101001001001_2

4. 110000111010_2, 100010_5 and 6072_8

5. 111101101101111_2, 2002344_5 and 75557_8

Review Exercise 3

1. (i) 486_{10} and 746_8
(ii) 3092_{10} and 6024_8
(iii) 16954_{10} and 41072_8

2. (i) 65345 (ii) 9867
(iii) 369635 (iv) 230400

3. (i) 10000101111_2, 13241_5 and 2057_8
(ii) 110001_2 , 144_5 and 61_8
(iii) 1110110100111100_2 , 3420412_5 and 166474_8

4. (i) 130_5 and 50_8 (ii) 312_5 and 122_8
(iii) 42_{10} and 52_8 (iv) 50_{10} and 200_5
(v) 453_{10} and 3303_5

5. (i) ten (ii) eight (iii) five
(iv) five (v) eight (vi) ten

6. (i) $4\times8^0=4\times1=4$ and $4\times10^0=4\times1=4$
(ii) $3\times5^1=3\times5=15$ and $3\times10^1=30$
(iii) $0\times8^2=0$ and $0\times10^2=0$
(iv) $2\times5^3=250$ and $2\times8^3=1024$
(v) $1\times5^4=625$ and $1\times10^4=10000$

Revision Exercise iv

1. (i) 8 (ii) 22 (iii) 9 (iv) 5.4
2. 33 **3.** Rs 2,700
4. 52.6 metres **5.** 122 days
6. 84 minutes **7.** 60 metres
8. 150 men **9.** 15 days
10. 40 persons

Exercise 4.1

1. 12 rupees **2.** 9,000 men
3. Rs 18,864 **4.** 200 cycles
5. 40 days **6.** 4 hours
7. 15 persons **8.** 4,000 kg
9. 24 days **10.** 8 hours

Exercise 4.2

1. Marium's share in the profit is Rs 9,000
Arifa's share in the profit is Rs 12,000

2. Share of Akram in the profit is Rs 16,800
Aslam's share in the profit is Rs 7,200
Asghar's share in the profit is Rs 4,800

3. Profit of Ali is Rs 15,000
Profit of Zain is Rs 10,500
Profit of Saad is Rs 12,000

4. (ii) 2:3 ; Rs 24,000 ; Rs 36,000
(iii) 1:2 ; Rs 20,000 ; Rs 40,000
(iv) 4:1 ; Rs 48,000 ; Rs 12,000
(v) Rs 75,000 ; Rs 24,000 ; Rs 36,000

5. Share of a daughter is Rs 180,000
Share of a son is Rs 360,000

6. Share of each widow is Rs 100,000
Share of each Son is Rs 560,000
Share of daughter is Rs 280,000

7. Share of mother is Rs 240,000
Share of widow is Rs 180,000
Share of each son is Rs 204,000
Share of each daughter is Rs 102,000

8. Share of each son is Rs 100,000

9. (ii) Rs 320,000 ; Rs 160,000
(iii) 4; 4; Rs 200,000 ; Rs 100,000
(iv) 1; 6; Rs 200,000 ; Rs 100,000
(v) 4; 6; Rs 160,000 ; Rs 80,000
(vi) 1; 6; Rs 200,000 ; Rs 100,000

Exercise 4.3

1. (i) 5,538.5 Yuan
(ii) 31250 Indian rupees
(iii) 15290 PKR (iv) 31,500 PKR

(v) 1500 Turkish Lira

2. 176,475 PKR **3.** 1,000 US dollar

4. 30,000 Indian rupees

5. 76,800 PKR

6. Cost of imported Fridge = Rs 74,220
Saving = Rs 3,280

Exercise 4.4

1. Rs.15000 **2.** Rs 8,000

3. Rs 75,000 **4.** 3%

5. 2.56 years **6.** 5%

7. 12.5 years

8. (i) Rs 4,000 (ii) Rs 38,220
(iii) 10 years (iv) Rs 5,000
(v) 30%

9. (i) Rs 1,666.60 (ii) Rs 60,000

10. Rs. 1,784,000

11. (i) Rs 8,000 ; Rs 232,000
(ii) Rs 7,333.30 ; Rs 216,000
(iii) Rs 4,666.60 ; Rs 264,000
(iv) Rs 6,000 ; Rs 224,000

Exercise 4.5

1. (i) Rs 105; 10% (ii) Rs 1,998; 20%
(iii) Rs 7,320; Rs 1,320 (iv) 1000,250

2. 8.33% profit **3.** 10% profit

4. 10% profit

5. (i) Rs 1,925 ; Rs 8,075 ; 19.25%
(ii) Rs 1,925 ; Rs 8,075 ; 19.25%
(iii) Rs 2,350 ; Rs 7,650 ; 23.50%
(iv) Rs 2,784 ; Rs 7,216 ; 27.84%

6. Rs 7,267.50 **7.** 31.6%

8. Rs 7,500

Exercise 4.6

1. (i) Rs 4,950 ; Rs 2,574 ;
Rs 1,336.50 ; Rs 445.50
(ii) Rs 7,425 ; Rs 3,861 ;
Rs 2,004.70 ; Rs 668.20
(iii) Rs 14,850 ; Rs 7,722 ;
Rs 4009.50 ; Rs 1,336.50

2. Rs 16,350 **3.** Rs 28,525

4. (ii) Rs 9,000 ; Rs 8100 ;
Rs 27,100
(iii) Rs 12,000 ; Rs 10,800 ;
Rs 9,720 ; Rs 32,520
(iv) Rs 500,000 ; Rs 20,000 ;
Rs 18,000 ; Rs 54,200
(v) Rs 1,000 ; Rs 40,000 ;
Rs 32,400 ; Rs 108,400

Exercise 4.7

1. Rs 29,500 **2.** Rs 1100

3. Rs 1,251,900 **4.** Rs 222,000

5. Rs 300,000 **6.** Rs 2,900,000

7. (ii) 2 ; 0 ; Rs 40,000 ;
80 ; 2% ; Rs 1,600
(iii) 4 ; Rs 750,000 ;
50 ; 10% ; Rs 29,500
(iv) Rs 597,000; Rs 4,000,000;
Rs 2,000 ; Rs 1,137,000

Revision Exercise 4

1. (i) c (ii) d (iii) c
(iv) b (v) a (vi) c
(vii) c (viii) b (ix) d (x) d

2. A gets Rs 27,000, B gets Rs 36,000
Total profit = Rs 162,000

3. 160 days

4. A single discount of 25%

5. Rs 1,000,000 **6.** Rs 960,000

7. 10%

Exercise 5.1

1.

S.No	Constants	Variables	Literals
(i)	3, 1	x	x
(ii)	0	–	–

(iii)	$2, \frac{1}{3}, -3, a, b$	–	a, b
(iv)	$3, -2, l, m$	–	l, m
(v)	$-2, 4, a, b, c$	–	a, b, c
(vi)	p, q, r	x, y	p,q,r,x,y
(vii)	$-5, 9, -4$	x, y	x, y
(viii)	$3^2, -2^2, p, q$	–	p, q
(ix)	$1, -1$	y	y
(x)	$\sqrt{3}, \sqrt{5}, -9, a, b$	–	a, b
(xi)	$7, -2, 3$	x	x
(xii)	-8	–	–

2. (i), (iv), (v), (vi), (vii), (ix), (xi), and (xii) are polynomials

3. (i) 0 (ii) 1 (iii) 2 (iv) 4
(v) 7 (vi) 1 (vii) 5 (viii) 3
(ix) 6 (x) 4 (xi) 0 (xii) 4

4. (i) 3 (ii) $\sqrt{4}$ (iii) –1 (iv) 1, 1
(v) 5,9,1 (vi) $\sqrt{2}, -\sqrt{3}$ (vii) –2
(viii) $-\frac{3}{4}$

5. (i) 2 (ii) 3 (iii) 2 (iv) 2
(v) 2 (vi) 1 (vii) 0 (viii) 3

6. (i) linear (ii) linear (iii) quadratic
(iv) quadratic (v) cubic
(vi) bi-quadratic (vii) cubic
(viii) bi-quadratic (ix) cubic
(x) bi-quadratic (xi) quadratic
(xii) bi-quadratic

Exercise 5.2

1. (i) $2x + z$
(ii) $5x + 8y^3 + 5xy - 2xy^3$
(iii) $x^2 - 4x - 1$
(iv) $-4ab + 2a^2 - 2b^2 - 1$

2. (i) $-6x - 3y + 2z$
(ii) $-7y^5 + y^4 + 6y^3 + 12y^2 + 8$
(iii) $x^2 + y^2 + z^2 - 15x + 15y - 9z$
(iv) $8y^2 + yz - 8z^2$

3. $10p^2 - 11pq + q^2$ **4.** $6a^3 + 3a + 7b$

5. (i) $-3x + 12y - 57xy$
(ii) $-7x - 2y + 9xy$
(iii) $x^3 + 6x^2y + 13xy^2 + 36y^3$
(iv) $x^4 + x^3y + xy^3 + y^4$

6. (i) $x\sqrt{x} + y\sqrt{y}$ (ii) $x^6 + 2x^3y^3 + y^6 - x^2y^2$
(iii) $a^4 - 2a^3b + 2ab^3 - b^4$
(iv) $a^3 + b^3 + c^3 - 3abc$

7. (i) $x^2 - 2x + 4$ (ii) $2x^2 - 8x + 4$
(iii) $x - 2$ (iv) $3x - 4y$
(v) $a^2 - ab + b^2$ (vi) $a^4 + a^2b^2 + b^4$

8. $x^2 + xy - y^2$

9. 47 **10.** $\frac{12 - x}{2}$

Review Exercise 5

1. (i) A polynomial with degree 3
Example: $2x^3 + 6x^2 - 3x + 5$
(ii) Symbol used to represent every element of a non-empty set.
In $2x + 3y$, variables are x and y.
(iii) a, b, c (iv) 2
(v) Highest degree of all the terms.
In $x^2 + 5x + 6$, the degree of polynomial is 2.

2. (i) d (ii) c (iii) a (iv) d (v) b
(vi) c (vii) d (viii) c (ix) d (x) a

3. (i) $-3a + 28b$ (ii) 10 (iii) 1
(iv) $\frac{910}{27}$ (v) 0

4. (i) $\frac{34x - 5y - 21z}{30}$ (ii) $x - y^2$
(iii) $x^2 - y^2 - 2y - 4$
(iv) $a^2 - 2a + 2$ (v) 1 **5.** –42

Exercise 6.1

1. (i) 1157625 (ii) 9216
(iii) 3249 (iv) 2704
(v) 9984 (vi) 2491

(vii) 9951 (viii) 9879

2. (i) 1.0609 (ii) 0.9801
(iii) 1.1025 (iv) 0.8281
(v) 0.9991 (vi) 24.9996

3. (i) 23 and 527 (ii) 14 and 194
(iii) 47 and 2207

4. (i) 11 and 119 (ii) 2.04 and 2.1616
(iii) 38 and 1442

Exercise 6.2

A. **1.** $4(x+2z)$ **2.** $5(x+2y+6z)$
3. $2x(1-2y+4z)$ **4.** $2a^2(1+5a-10a^2)$
5. $ab(3a+7b-8ab)$
6. $6abc(a+2b-6c)$
7. $(x+2y)(5+3z)$ **8.** $(c-d)(ab+x)$
9. $(x+5)(x+6y)$ **10.** $(x+2z)(7y-5a)$
11. $3(1+2z)(x^2+2y^2)$
12. $(x-z)(x-7y)$

B. **1.** $(a+5)^2$ **2.** $(x+6y)^2$
3. $(2x+3y)^2$ **4.** $(4a+5b)^2$
5. $(b-c)^2$ **6.** $(7p-1)^2$
7. $(9c-2d)^2$ **8.** $(12x^2-3y^2)^2$
9. $3(a-b)^2$ **10.** $(\frac{1}{2}x-\frac{1}{3}y)^2$

C. **1.** $(9x+2y)(9x-2y)$
2. $(13a+10b)(13a-10b)$
3. $3(a+3b^2)(a-3b^2)$
4. $2(p+3q)(p-3q)$
5. $5(x+5y)(x-5y)$
6. $(\frac{5}{x}+\frac{y}{4})(\frac{5}{x}-\frac{y}{4})$
7. $(\frac{x}{12}+y)(\frac{x}{12}-y)$
8. $(\frac{6}{5}l+\frac{7}{2}d)(\frac{6}{5}l-\frac{7}{2}d)$
9. $(c+a-b)(c-a+b)$
10. $(x+y+z)(x+y-z)$
11. $(a+b+p+q)(a+b-p-q)$
12. $(12+y^2)(12-y^2)$
13. $2(4a+5b)(4a-5b)$
14. $(x^2+y^2)(x+y)(x-y)$
15. $(2a+b+3c)(2a+b-3c)$

D. **1.** $(x+y+z)(x+y-z)$
2. $(a-b+c)(a-b-c)$
3. $(8a+3b+c)(8a+3b-c)$
4. $(2p-3q+7r)(2p-3q-7r)$
5. $(x^2+y^2+5z)(x^2+y^2-5z)$
6. $(\frac{x}{2}-y+\frac{c}{6})(\frac{x}{2}-y-\frac{c}{6})$
7. $2(a+b+c)(a+b-c)$
8. $(9c+d+4p)(9c+d-4p)$
9. $(2x+3+4y)(2x+3-4y)$
10. $(6+5a-7b)(6-5a+7b)$

Exercise 6.3

1. (i) $8x^3+36x^2+54x+27$
(ii) $125x^3-75x^2+15x-1$
(iii) $8x^3-36x^2y+54xy^2-27y^3$
(iv) $8a^3+60a^2b+150ab^2+125b^3$
(v) $x^3+21x^2z+147xz^2+343z^3$
(vi) $\frac{x^3}{8}-\frac{3x^2}{4}+\frac{3x}{2}-1$
(vii) $125+225a+135a^2+27a^3$
(viii) $125-\frac{75a}{4}+\frac{15a^2}{16}-\frac{a^3}{64}$
(xi) $125a^3-225a^2b+135ab^2-27b^3$
(x) $\frac{x^3}{27}+\frac{x^2y}{15}+\frac{xy^2}{25}+\frac{y^3}{125}$

2. (i) 5832 (ii) 2197
(iii) 1157625 (iv) 1030.301
(v) 0.941192 (vi) 11.390625

3. (i) 198 (ii) – 110

(iii) $-1\frac{3}{8}$ (iv) 1000.001

4. (i) 140 (ii) – 14

(iii) $1\frac{1}{27}$ (iv) 999.999

Exercise 6.4

1. Linear equations in one variable:
(ii), (v)
Linear equations in two variables:
(i), (vi)
Simultaneous linear equations:
(iii), (iv)

2. (i) $x + y = 50$ (ii) $6x = 3y$

(iii) $2x + 5 = 25$ (iv) $3x - 6y = 45$

(v) $x = \frac{2y}{3}$ (vi) (6, 0)

Exercise 6.5

1. (i) {(3,4)} (ii) $\{(\frac{7}{8}, 1\frac{3}{8})\}$

(iii) { (2, 3)} (iv) $\{(\frac{1}{2}, \frac{1}{3})\}$

(v) $\{(1\frac{1}{2}, 5)\}$ (vi) $\{(-2\frac{3}{5}, 6\frac{2}{5})\}$

2. (i) {(5, 2)} (ii) {(–1, 2)}

(iii) {(–2, –3)} (iv) {(–1, –2)}

(v) $\{(\frac{1}{3}, \frac{2}{3})\}$ (vi) {(5, 2)}

3. (i) {(1, 1)} (ii) {(–5, –1)}

(iii) $\{(4\frac{2}{5}, 1\frac{1}{5})\}$ (iv) {(6 1)}

(v) { } (vi) {(1, –1)}

Exercise 6.7

1. (i) $y = 0$ (ii) $27y - 5 = 0$

(iii) $5y - 1 = 0$ (iv) $-ay + b = 0$

(v) $7y - 6 = 0$ (vi) $17y - 16 = 0$

2. (i) $S = V_f t - \frac{1}{2}at^2$ (ii) $S = V_f t - \frac{3}{2}gt^2$

(iii) $2V_f t - gt^2 = 2S$

3. (i) $4a^2 = 4b^2 + 4b + 5$
(ii) $4a^2 - 9b^2 + 4 = 0$
(iii) $q^4 = p^4 - 4p^2$
(iv) $p^4 - q^4 = 2$
(v) $3y^2 + 8y - 7 = 0$
(vi) $m^2 - n^2 - 4 = 0$

Review Exercise 6

1. (i) (b) (ii) (d) (iii) (c) (iv) (c)
(v) (b)

2. for $x - \frac{1}{x} = 5$ (i) 27 (ii) 727

(iii) 140 (iv) 19602

for $x + \frac{1}{x} = 3$ (i) 7 (ii) 47

(iii) 18 (iv) 322

4. (i) $5(x + 2y + 3z)$ (ii) $(x + 7)^2$
(iii) $(12x + 11y)(12x - 11y)$
(iv) $(5x - y)^2$
(v) $(a + 2b + 3y)(a + 2b - 3y)$
(vi) $(1 + \frac{1}{x^2})(1 + \frac{1}{x})(1 - \frac{1}{x})$

5. (i) $8x^3 + 36x^2y + 54xy^2 + 27y^3$
(ii) $27x^3 - 108x^2y + 144xy^2 - 64y^3$
(iii) $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

(iv) $x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$

7. (a) (i) {(4, 2)} (ii) $\{(\frac{22}{9}, \frac{-7}{9})\}$

(b) (i) {(1, 2)} (ii) {(5, 4)}
(c) (i) {(2, 1) (ii) {(11, 1)}

8. 32 cm^2
9. (i) $21k^2 + 4k - 5 = 0$ (ii) $m^2 + 2 = n^2$

Exercise 7.1

1. (*a*) 120° (b) 135° (c) 120° (d) 80°
2. $m\angle 1 = 65^\circ = m\angle 5 = m\angle 7$;
$m\angle 2 = m\angle 4 = m\angle 6 = m\angle 8 = 115^\circ$;
3. $m\angle APE = 130^\circ = m\angle BPQ = m\angle DQF = m\angle PQC$;
$m\angle APQ = 50^\circ = m\angle CQF = m\angle BPE$
4. (i) $\angle AGE$ (ii) $\angle BGH$ (iii) $\angle DHF$
(iv) (*a*) $\angle AGH$ (b) $\angle GHD$ (c) $\angle CHF$
5. (i) $x = 11^\circ$ (ii) $x = 5^\circ$, $y = 25^\circ$, $z = 130^\circ$
(iii) $y = 135^\circ$, $z = 105^\circ$
(iv) $x = 80^\circ$, $y = 100^\circ$, $z = 80^\circ$

Exercise 7.2

1. (i) T (ii) T (iii) T (iv) T (v) T
(vi) T (vii) T (viii) T (ix) T (x) T
2. L = 6 cm, B = 3 cm
3. Angles are 65°, 115°, 65°, 115°
4. $m\angle W = 60^\circ = m\angle Y$
$m\angle X = 120^\circ = m\angle Z$
5. 6 : 3, **6.** 2:1
7. P. lies inside the circle, Q outside and R lies on the circle.
8. All statements are False.
Go through the definition for correct statements.

Review Exercise 7

1. (i) Coplanar (ii) Parallel
(iii) Mid-point, third
(iv) (b) alternate (c) Supplementary
Same side, Transversal
(v) (*a*) Corresponding, alternate
(b) Angles, side, supplementary
(vi) Plane, three, more
(vii) Parallel, Congruent
(viii) Parallelogram (ix) Interior
(x) Regular polygon (xi) Circle
(xii) Chord
2. $m\angle Q = 110^\circ = m\angle S$, $m\angle R = 70^\circ$,
$m\ \overline{AD} = 4.5$, $m\ \overline{CD} = 6$cm
$m\angle A = m\angle C = 70^\circ$,
$m\angle B = m\angle D = 110^\circ$. No.
3. $m\angle A = 65^\circ = m\angle BCD$,
$m\angle B = 115^\circ = m\angle BCF = m\angle ADC$.
(i) ($\angle A$, $\angle EDC$); ($\angle ADC$, $\angle BCF$)
(ii) ($\angle EDC$, $\angle BCD$); ($\angle B$, $\angle BCF$)
(iii) ($\angle A$, $\angle B$); ($\angle A$, $\angle ADC$);
($\angle ADC$, $\angle BCD$); ($\angle ADC$, $\angle CDE$);
($\angle B$, $\angle BCD$) and ($\angle BCD$, $\angle BCF$)
6. (i) 8 cm (ii) 2.5 cm, 3.5 cm, 5 cm
(iii) (a) One (b) One

Review Exercise 8

(i) (b) (ii) (b) (iii) (a) (iv) (d)
(v) (c) (vi) (c) (vii) (c) (viii) (b)

Exercise 9.2

1. (i) 15 cm (ii) 20 cm (iii) 24 cm
(iv) 7.5 cm (v) 6 cm (vi) 4.8 cm
(vii) 3.5 cm (viii) 7.2 cm
2. (ii) and (v)
3. (i) 5 cm (ii) 6 cm (iii) 7 cm
(iv) 8 cm (v) 1.1 cm (vi) 2.2 cm
(vii) 1.5cm (viii) 5.5cm (ix) 7.5cm
4. 3.9m **5.** 16m
6. (i) 8.5 cm (ii) 55 cm
(iii) 7 cm (iv) 8 cm
7. 600 m **8.** 3072m^2

9. Fig (i) $x = 24$ cm, $y = 40$ cm
Fig (ii) $x = 15$ cm, $y = 20$ cm
Fig (iii) $x = 2\sqrt{337}$ cm, $y = 40$ cm

Exercise 9.3

1. (i) 124.89 cm^2 (ii) 84 cm^2
(iii) 24 cm^2 (iv) 176 cm^2
(v) 104.88 cm^2

2. (i) 48 cm^2 (ii) 71.5 cm^2
(iii) 40.24cm^2 (iv) 265.33 cm^2

3. $\frac{\sqrt{3}a^2}{4}$ cm^2

4. (i) c =12 cm, ▲ABD= 74.8cm^2
(ii) f = 3.9 cm, ▲DEF= 6.73cm^2
(iii) n = 6.4 cm, ▲LMN= 19.04cm^2
(iv) r = 5 cm, ▲PQR= 9.35cm^2

Exercise 9.4

1. 61.68 cm^2 **2.** 12 cm^2
3. 199.35 **4.** 6549.64 m^2
5. 114 cm^2

Exercise 9.5

1. (i) 15393 cm^2 (ii) 14 cm
(iii) 2.21 cm^2 (iv) 4.18 cm^2
(v) 12.06 cm^2

2. (i) 7 cm (ii) 14 cm
(iii) 84 cm (iv) 8.8 cm
(v) 3.5 cm

3. (i) 38792 cm^3 (ii) 492807 cm^3
(iii) 91.9 cm^3 (iv) 22449297.5 cm^3
(v) 61.6 cm^3

4. (i) 1436 litres (ii) 4849 litres
(iii) 91952 litres (iv) 22449 litres
(v) 61600 litres

5. (i) 0.01149667 m^3
(ii) 0.09198933 m^3, 19.9 letres
or 115 l (approx)
(iii) 15.293 m^3 (v) 4.851 m^3
15293 l 4851 litre
(iv) 26.6197 m^3
26619.7 litre

6. (a) A = four lines increases
(b) V= Eight time increases

7. (a) 1:9 (b) 1:27

8. (i) 2009947 m^3
(ii) 478646

9. 9 m^3 **10.** 7241.14 cm^3

Exercise 9.6

1. (i) S = 35 cm, Base area = 1386 cm^2,
covered surface area = 220 cm^2
Total Surface Area (T.S.A) = 3696 cm^2
(ii) r = 42 cm, B.A = 5544 cm^2,
C.S.A = 9240 cm^2, C.S.A = 14784 cm^2
(iii) h = 48 cm, B.A 4073.14 cm^2,
(C.S.A) 6788.57 cm^2,
T.S.A = 10861.71 cm^2
(iv) r = 54.258 cm, B.A = 9252.35 cm^2,
C.S.A = 17052.51429 cm^2,
T.S.A = 26304.86 cm^2

2. (i) S = 7.615 cm, V = 22 cm^3
(ii) S = 6.946 cm, V = 22 cm^3
(iii) r = 21 cm, V = 616 cm^3
(iv) r = 10.5 cm, S = 17.5 cm
(v) h = 104.810 cm, V = 691.746 cm^3

Exercise 9.7

1. 1436 cm^3 **2.** 396cm^3
3. 137.9 m^3 **4.** 2640 kg
5. 66 m^3 **6.** 465.69 m^3 **7.** $\frac{3}{4}$cm

Review Exercise 9

1. c **2.** b **3.** b **4.** c
5. d **6.** d **7.** a **8.** a
9. b **10.** d

ANSWERS

Exercise 10.1

1. (a) branch, theorems, proved, logical reasoning them (or these)
(b) Inductive, Deductive
(c) Inductive (d) Deductive
(e) Accepted without proof
(f) Axioms, Postulates
(g) Postulates
(h) Undefined Terms, Assumptions or Fundamental Agreements, Defined Terms, Propositions (Theorems)

2. (a) False (b) True (c) False
(d) True (e) True (f) True

Exercise 10.2

1. (i) $\angle AOC$ (ii) $\angle POC$ (iii) 40°, 130°

3. (a) 140°, 40° (b) 135°, 45°, 135°

6. $\angle AOE$, $\angle BOD$; $\angle BOD$, $\angle COE$; $\angle DOC$, $\angle COE$; $\angle DOC$, $\angle AOE$

Exercise 10.3

6. (i) (a) $\angle c$, $\angle d$; $\angle g$, $\angle h$
(b) $\angle a$, $\angle b$; $\angle e$, $\angle f$; $\angle f$, $\angle a$
(c) $\angle a$, $\angle e$; $\angle b$, $\angle f$
(ii) (a) $\angle c$, $\angle e$; $\angle d$, $\angle f$; $\angle c$, $\angle d$; $\angle e$, $\angle f$
(b) $\angle a$, $\angle b$; $\angle g$, $\angle h$;
(c) $\angle c$, $\angle f$; $\angle d$, $\angle e$
(iii) (a) $\angle c$, $\angle d$
(b) $\angle e$, $\angle h$; $\angle h$, $\angle g$; $\angle g$, $\angle f$; $\angle f$, $\angle e$
(c) $\angle e$, $\angle g$; $\angle h$, $\angle f$

Exercise 10.5

1. (a) 90° (b) 120° [180 – (135–75)]
(c) 55° (d) 140° [180–115+(180–105)]
(e) 65°

Review Exercise 10

8. (a) (i) F (ii) T (iii) F (iv) F (v) T
(b) (i) Plane (ii) Line
(iii) Perpendicular
(iv) Alternate and corresponding angles
(v) Sum

9. (i) Lines AB,CD; AB,FG; FG,CD; EF,GC
(ii) (a) 22 (b) 75+50=125 (c) 20

Exercise 11.1

1. $\text{Sin m}\angle P = \text{Cos m}\angle Q = \frac{3}{5}$,

$\text{Cos m}\angle P = \text{Sin m}\angle Q = \frac{4}{5}$,

$\text{Tan m}\angle P = \text{Cot m}\angle Q = \frac{3}{4}$,

$\text{Cot m}\angle P = \text{Tan m}\angle Q = \frac{4}{3}$,

$\text{Cosec m}\angle P = \text{Sec m}\angle Q = \frac{5}{3}$,

$\text{Cosec m}\angle Q = \text{Sec m}\angle P = \frac{5}{4}$

2. (i) $\frac{4}{5}$ (ii) $\frac{4}{5}$ (iii) $\frac{4}{3}$ (iv) $\frac{5}{3}$
(v) $\frac{5}{3}$ (vi) $\frac{4}{3}$ (vii) $\frac{3}{5}$ (viii) $\frac{3}{5}$
(ix) $\frac{3}{4}$ (x) $\frac{5}{4}$ (xi) $\frac{5}{4}$ (xii) $\frac{3}{4}$;
Cos x = Sin y, Sec x = Cosec y,
Tan x = Cot y, Cot x = Tan y,
Cosec x = Sec y

Exercise 11.2

1. (i) 30° (ii) 45° (iii) 60°

2. (i) 60° (ii) 45° (iii) 30°

3. (i) 30° (ii) 45° (iii) 60°

4. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{13}{3}$ (iii) $\frac{19\sqrt{3}}{12}$

(iv) 0 (v) $-\frac{1}{6}$ (vi) $3\frac{1}{3}$ (vii) 0

5. (i) 60° (ii) 30° (iii) 45°
(iv) 20° (v) 50° (vi) 35°
(vii) 80° (viii) 25° (ix) 55°

6. (i) 55° (ii) 25° (iii) 70°
(iv) 37° (v) 34° (vi) 15°

Exercise 11.3

A. 1. $m\angle B=60^\circ$, $m\overline{AB}=6$, $m\overline{AC}=3\sqrt{3}$
2. $m\angle F=30^\circ$, $m\overline{EF}=4\sqrt{3}$, $m\overline{DF}=8$
3. $m\angle L=45^\circ$, $m\overline{LN}=8\sqrt{2}$, $m\overline{MN}=8$ cm

B. 1. $m\angle B=60^\circ$, $m\overline{BC}=5$ cm,
$m\overline{AC}=5\sqrt{3}$ cm
2. $m\angle D=30^\circ$, $m\overline{BC}=7$ cm,
$m\overline{DC}=7\sqrt{3}$ cm
3. $m\angle G=45^\circ$, $m\overline{GC}=8$ cm,
$m\overline{CD}=8$ cm
4. $m\angle E=60^\circ$, $m\overline{EC}=4.5$ cm,
$m\overline{CF}=\frac{9\sqrt{3}}{2}$ cm
5. $m\angle M=30^\circ$, $m\overline{ON}=8$ cm,
$m\overline{OM}=8\sqrt{3}$ cm
6. $m\angle R=30^\circ$, $m\overline{PQ}=3$ cm,
$m\overline{PR}=6$ cm

C. 1. $m\angle L=30^\circ$, $m\angle N=60^\circ$, $m=8$ cm
2. $m\angle P=60^\circ$, $m\angle R=30^\circ$, $q=12$ cm
3. $m\angle X=45^\circ$, $m\angle Z=45^\circ$, $y=7\sqrt{2}$ cm

D. 1. $m\angle A=30^\circ$, $m\angle B=60^\circ$, $b=5\sqrt{3}$ cm
2. $m\angle E=30^\circ$, $m\angle D=60^\circ$, $e=7$ cm
3. $m\angle M=45^\circ$, $m\angle N=45^\circ$, $n=8$ cm

Exercise 11.4

1. 6 m **2.** 200 m **3.** 294 m (approx)
4. 11.55 m (approx) **5.** 50 m

Review Exercise 11

2. 9 m **3.** 120 m

4. (i) Sec 60° (ii) Tan 30° (iii) Sin 45°
(iv) Cosec 60° (v) Cot $m\angle C$
(vi) $(90^\circ - m\angle A)$ (vii) $m\angle A$
(viii) 60° (ix) 80° (x) θ

5. (i) (b) (ii) (c) (iii) (c) (iv) (a)
(v) (b) (vi) (c) (vii) (b)

6. (i) F (ii) F (iii) F (iv) T
(v) T (vi) F (vii) T (viii) F
(ix) F (x) F

Revision Exercise 12

1.

Class No.	Frequency	Class Interval	Class Boundaries	Upper Class limit	Class width
1st	03	24-28	23.5-28.5	28.5	5
2nd	16	29-33	28.5-33.5	33.5	5
3rd	12	34-38	33.5-38.5	38.5	5
4th	23	39-43	38.5-43.5	43.5	5
5th	16	44-48	43.5-48.5	48.5	5

2 (i)

Marks Obtained	0	1	3	4	5	6
No. of students	4	2	9	5	4	1

(ii) Highest mark = 6 (iii) 3 (iv) 6 (v) Range = 6

3. (i) Frequency Chart

Hours	1	2	3	4	5	6
No. of members	5	8	7	4	2	1

(ii) 27 (iii) 6 hours

Exercise 12.2

1. Frequency Distribution

Weight (kg)	29-32	33-36	37-40	41-44	45-47
No. of scouts	4	6	8	8	4

Students may draw the histogram themselves from 1 to 6.

7. (i) 5 or > 5 (ii) 5 or < 5 **8.** (iv) Except 1, 2 and 11

Exercise 12.3

1. Rs 40 per kg 2. Mean

2. 2 times 2 times 2 times 2 times 3 times

Mean = , Median = _____, Mode = _____, Incorrect

3. Mean = , Median = _____, Mode _____, Incorrect

4. Weighted means of marks secured in Maths = $\frac{410}{9}$

Weighted means of marks secured in science = $\frac{434}{9}$

Thus the result of science subject is better than Math's

5. Size of family = 3.76 persons per family

Review Exercise 12

2. Frequency Table

Quantity of water per family	
No. of families	

Class interval = _____

Students may complete the frequency table themselves from Q2 to Q4.

6. (a) The area increase by 4 times (b) The volume increase by 8 times

7. (a) 1 : 9 (b) 1 : 27 **10.** 7241.142 cm^3

INDEX