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THE TEXTBOOK OF MATHEMATICS

For Class – IX

Sindh Textbook Board

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PREFACE

The Sindh Textbook Board, is assigned with preparation and publication of the textbooks to equip our new generation with knowledge, skills and ability to face the challenges of new millennium in the fields of Science, Technology and Humanities. The textbooks are also aimed at inculcating the ingredients of universal brotherhood to reflect the valiant deeds of our forebears and portray the illuminating patterns of our rich cultural heritage and tradition.

The new editions include introductory paragraphs, information boxes, summaries and a variety of extensive exercises which I think will not only develop the interest but also add a lot to the utility of the book.

The Sindh Textbook Board has taken great pains and incurred expenditure in publishing this book inspite of its limitations. A textbook is indeed not the last word and there is always room for improvement. While the authors have tried their level best to make the most suitable presentation, both in terms of concept and treatment, there may still have some deficiencies and omissions. Learned teachers and worthy students are, therefore, requested to be kind enough to point out the shortcomings of the text or diagrams and to communicate their suggestions and objections for the improvement of the next edition of this book.

In the end, I am thankful to Association For Academic Quality (AFAQ), our learned authors, editors and specialist of Board for their relentless service rendered for the cause of education.

Chairman
Sindh Textbook Board

Unit 1

• Weightage = 7%

REAL AND COMPLEX NUMBERS

Student Learning Outcomes (SLOs)

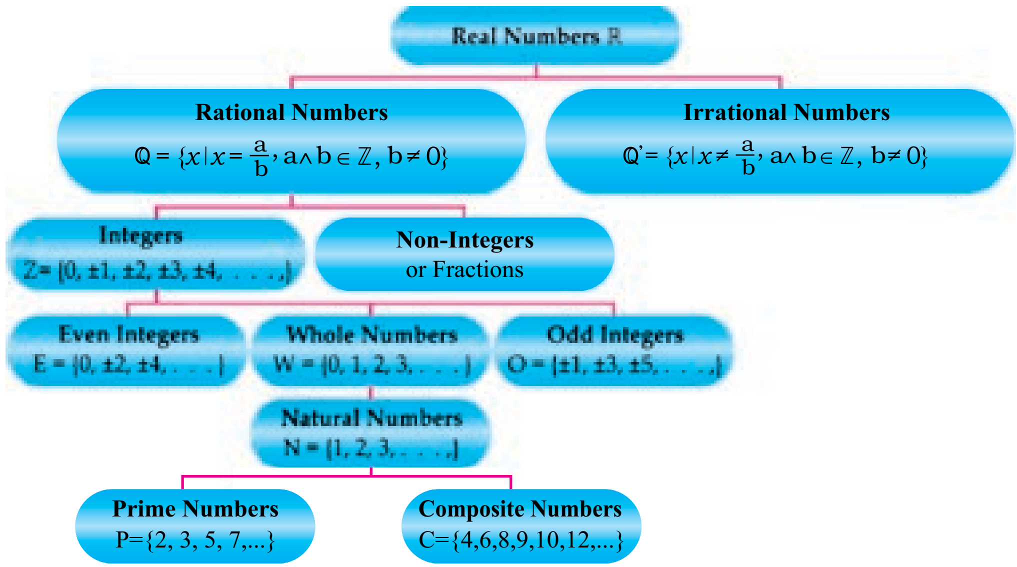
After completing this unit, students will be able to:

- Recall the set of real numbers as the union set of rational and irrational numbers.
- Represent real numbers on the number line.
- Demonstrate a number with terminating and non–terminating recurring decimal on the number line.
- Distinguish the decimal representation of rational and irrational numbers.
- Know the properties of real numbers
- Identify radicals and radicands.
- Differentiate between radical and exponential forms of an expression.
- Transform an expression given in radical form to an exponent form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.
- Elucidate, then define a complex number z represented by an expression of the form (a, b) or $z = a + i\,b$, where a is real and b is imaginary part and here $i = \sqrt{-1}$
- Recognize ***a*** as real part and ***b*** as imaginary part of $z = a + ib$ or $z = (a, b)$
- Define conjugate of a complex number
- Know the condition of equality of complex numbers.
- Carry–out basic operations (i.e. addition, subtraction, multiplication and division) on complex numbers.

Introduction

In pervious classes we have learned various kinds of numbers such as natural numbers (counting numbers), whole numbers, integers, rational numbers etc.

All these numbers are contained in the set of real numbers. Hence classification of real numbers is given below:



1.1 Real Numbers

1.1.1 Recall the set of real numbers as the union set of rational and irrational numbers.

The set of real numbers is the union of the set of rational and irrational numbers. i.e., $\mathbb{R} = Q \cup Q'$

We have already learned about rational and irrational numbers. Real numbers have many properties as the properties of rational numbers.

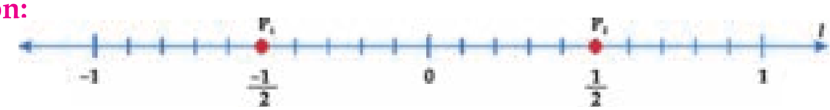
1.1.2 Represent Real Numbers on the Number Line

In the previous classes we have already studied whole numbers, integers and their representation on a number line. Similarly we can represent real numbers on number line.

Let us see the following examples.

Example 01 Represent the numbers $-\frac{1}{2}$ and $\frac{1}{2}$ on the number line l

Solution:

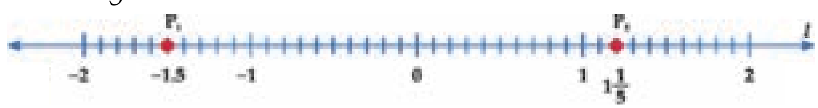


Thus, in the above figure the point P_1 represents number $-\frac{1}{2}$ and the point P_2 represents $\frac{1}{2}$.

Example 02 Represent –1.5 and $1\frac{1}{5}$ on the number line.

Solution:

Similar in the figure, point P_1 represents number –1.5 and P_2 represent number $1\frac{1}{5}$.



1.1.3 Demonstrate a Number with Terminating and Non–Terminating Recurring Decimal on the Number Line

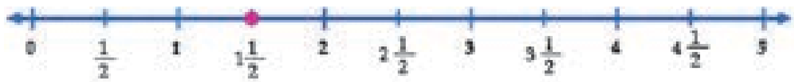
In order to locate a number with terminating or non–terminating and recurring decimal on the number line, the points associated with the rational numbers $\frac{a}{b}$ and where a, b are positive integers, we sub–divide each unit length into b equal parts. Then the a^{th} point of division to the right of the origin represents $\frac{a}{b}$ and that to the left of the origin at the same distance represents $-\frac{a}{b}$.

Example 01

Illustrate the following terminating decimal fractions on the number line.

i. $\frac{3}{2}$ ii. $\frac{5}{4}$

i. $\frac{3}{2}=1\frac{1}{2}$



ii. $\frac{5}{4}=1\frac{1}{4}$

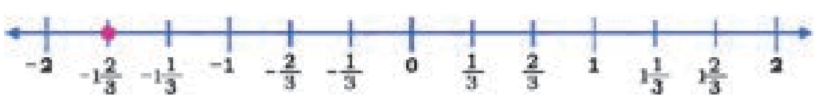
Example 02

Illustrate the following non-terminating and recurring decimal fractions on number line.

i. $\frac{11}{6}$ ii. $-\frac{5}{3}$

i. $\frac{11}{6}=1\frac{5}{6}$

iii. $-\frac{5}{3}=-1\frac{2}{3}$



1.1.4 Distinguish the Decimal Representation of Rational and Irrational Numbers.

When we represent rational numbers in the decimal form then two types of decimal fractions are possible i.e. terminating or non–terminating and recurring decimal fractions, while irrational numbers are represented as non- terminating and non-recurring decimal fraction. We represent them in below table.

S.No	Number	Remarks
1.	$\frac{1}{2} = 0.5$	Terminating decimal fraction
2.	$\frac{1}{4} = 0.25$	Terminating decimal fraction
3.	$\frac{1}{3} = 0.333...$	Non-terminating and recurring decimal fraction
4.	$\frac{9}{11} = 0.818181...$	Non-terminating and recurring decimal fraction
5.	$\sqrt{2} = 1.414213...$	Non-terminating and non-recurring decimal fraction
6.	$\sqrt{3} = 1.73205...$	Non-terminating and non-recurring decimal fraction

Exercise 1.1

1. Identify the following numbers as rational and irrational numbers and also write each one in separate column.

(i) $\frac{1}{5}$ **(ii)** $\frac{\sqrt{2}}{8}$ **(iii)** $\frac{5}{\sqrt{6}}$ **(iv)** $\frac{2}{8}$ **(v)** $\frac{1}{\sqrt{3}}$ **(vi)** $\sqrt{8}$

(vii) 0 **(viii)** π **(ix)** $\sqrt{5}$ **(x)** $\frac{22}{3}$ **(xi)** $\frac{1}{\pi}$ **(xii)** $\frac{11}{12}$

2. Convert the following into decimal fractions. Also indicate them as terminating and non–terminating decimal fractions.

(i) $\frac{5}{8}$ **(ii)** $\frac{4}{18}$ **(iii)** $\frac{1}{15}$ **(iv)** $\frac{49}{8}$ **(v)** $\frac{207}{15}$ **(vi)** $\frac{50}{76}$

3. Illustrate the following rational numbers on number line.

(i) $\frac{8}{10}$ (ii) $-\frac{8}{10}$ (iii) $1\frac{1}{4}$ (iv) $-1\frac{1}{4}$ (v) $\frac{2}{3}$ (vi) $-\frac{2}{3}$

4. Can you make a list of all real numbers between 1 and 2?

5. Give reason, why pi (π) is an irrational number?

6. Tick (✓) the correct statements.

(i) $\frac{5}{7}$ is an example of irrational number.

(ii) π is an irrational number.

(iii) 0.31591… is an example of non–terminating and non–recurring decimal fraction.

(iv) $0.12\overline{3}$ is an example of recurring decimal fraction.

(v) $\frac{1}{3}, \frac{2}{3}$ are lying between 0 and 1.

(vi) $\frac{1}{\sqrt{3}}$ is an example of rational number.

1.2 Properties of Real Numbers.

In real numbers there exist properties with respect to addition and multiplication. For real numbers a and b, the sum is $a + b$ and product is written as $a.b$ or $a \times b$ or simply ab.

1.2.1 Know the Properties of Real Numbers

(a) Properties of Real Numbers with respect to Addition

(i) Closure Property:

Sum of any two real numbers is again a real number.

i.e. $\forall a, b \in \mathbb{R} \Rightarrow a + b \in \mathbb{R}$ is called closure property w.r.t addition.

e.g. (i) $5, 7 \in \mathbb{R} \Rightarrow 5 + 7 = 12 \in \mathbb{R}$

(ii) $\frac{4}{5}, \frac{3}{4} \in \mathbb{R} \Rightarrow \frac{4}{5} + \frac{3}{4} = \frac{16+15}{20} = \frac{31}{4} \in \mathbb{R}$

(ii) Commutative Property:

For any two real numbers a and b

$$a+b=b+a$$

is called commutative property w.r.t addition

e.g. (i) $3+7=7+3$ (ii) $\sqrt{5}+\sqrt{6}=\sqrt{6}+\sqrt{5}$.

(iii) Associative Property:

For any three real numbers a , b and c such that

$$(a+b)+c=a+(b+c)$$

is called associative property w.r.t addition.

e.g. $(4+5)+6=4+(5+6)$

(iv) Additive Identity:

There exists a number $0\in\mathbb{R}$ such that

$$a+0=a=0+a, \quad \forall a\in\mathbb{R}$$

'0' is called additive identity

e.g. $3+0=3=0+3, \; \frac{7}{8}+0=\frac{7}{8}=0+\frac{7}{8},$ etc

(v) Additive Inverse:

For each $a\in\mathbb{R}$, there exist $-a\in\mathbb{R}$ such that $a+(-a)=0=(-a)+a$

so, $-a$ and a are additive inverses of each other.

e.g. $6+(-6)=0=(-6)+6=0$

Here 6 and –6 are additive inverses of each other.

(b) Properties of Real Numbers with respect to Multiplication

(i) Closure Property:

The product of any two real numbers a and b is again a real number. i.e., $a,b\in\mathbb{R}\Rightarrow ab\in\mathbb{R}$, is called closure properly w.r.t multiplication

e.g. (i) $5,7\in\mathbb{R}\Rightarrow(5)(7)=35\in\mathbb{R}$

(ii) $\frac{3}{5},\frac{6}{7}\in\mathbb{R}\Rightarrow\left(\frac{3}{5}\right)\left(\frac{6}{7}\right)=\frac{18}{35}\in\mathbb{R}$, etc

(ii) Commutative Property:

For any two real numbers a and b

$ab = ba$ is called commutative property w.r.t multiplication.

e.g. (i) $\sqrt{3}, \sqrt{5} \in \mathbb{R} \Rightarrow (\sqrt{3})(\sqrt{5}) = (\sqrt{5})(\sqrt{3})$

(ii) $3, 4 \in \mathbb{R} \Rightarrow 3\times 4 = 4\times 3$ etc.

(iii) Associative Property:

For any three real numbers a,b and c

$(ab)c = a(bc)$ is called associative property w.r.t multiplication.

e.g. (i) $4,5,6 \in \mathbb{R}$, then $(4\times 5)\times 6 = 4\times(5\times 6)$,

(ii) $\frac{2}{5}, 4, \sqrt{3} \in \mathbb{R}$, then $(\frac{2}{5}\times 4)\times\sqrt{3} = \frac{2}{5}\times(4\times\sqrt{3})$, etc.

(iv) Multiplicative Identity:

For any real number a there exist a number $1 \in \mathbb{R}$

$a\times 1 = 1\times a = a$, '1' is called multiplicative identity.

e.g. $1\times 3 = 3\times 1 = 3$, $\frac{3}{5}\times 1 = 1\times\frac{3}{5} = \frac{3}{5}$, etc.

(v) Multiplicative Inverse:

For each $a \in \mathbb{R} (a \neq 0)$ there exists an element $\frac{1}{a}$ or $a^{-1} \in \mathbb{R}$

$a\times\frac{1}{a} = \frac{1}{a}\times a = 1$, thus $\frac{1}{a}$ and a are the multiplicative inverses of each other.

e.g. $3\times\frac{1}{3} = 1 = \frac{1}{3}\times 3$

Here 3 and $\frac{1}{3}$ are multiplicative inverses of each other.

(c) Distributive Property of Multiplication over Addition

For any three real numbers a,b,c such that

(i) $a(b+c) = ab + ac$, it is called Distributive Property of multiplication over addition. (Left Distributive Property)

(ii) $(a+b)c = ac+bc$, it is called distributive property of multiplication over addition. (Right Distributive Property)

e.g. $3(5+7) = 3\times5 + 3\times7$, (Left Distributive Property)

$(3+7)2 = 3\times2 + 7\times2$, (Right Distributive Property)

Note: $a(b-c) = ab-ac$ is the left distributive property of multiplication over subtraction.

(d) Properties of Equality of Real Numbers

Following are the properties of equality of real numbers.

(i) Reflexive Property

If $a \in \mathbb{R}$ then $a = a$.

(ii) Symmetric Property

If $a,b \in \mathbb{R}$ then $a = b \Leftrightarrow b = a$.

(iii) Transitive Property

If $a,b,c \in \mathbb{R}$ then, $a = b$ and $b = c \Leftrightarrow a = c$.

(iv) Additive Property

If $a,b,c \in \mathbb{R}$then, $a = b \Leftrightarrow a+c = b+c$.

(v) Multiplicative Property

If $a,b,c \in \mathbb{R}$ such that, $a = b$ then $ac = bc$.

(vi) Cancellation Property for Addition

If $a,b,c \in \mathbb{R}$, if $a+c = b+c$ then $a = b$

(vii) Cancellation property for multiplication

If $a,b,c \in \mathbb{R}$ and $c \neq 0$ if $ac = bc$ then, $a = b$

(e) Properties of Inequalities of Real Numbers.

Following are the properties of inequalities of real numbers.

(i) Trichotomy Property

If $a,b,c \in \mathbb{R}$ then $a > b$ or $a < b$ or $a = b$.

(ii) Transitive Property

If $a,b,c \in \mathbb{R}$ then

(a) $a < b$ and $b < c \Rightarrow a < c$,

(b) $a > b$ and $b > c \Rightarrow a > c$.

(iii) Additive Property

If $a, b, c \in \mathbb{R}$ then

(a) $a < b \Rightarrow a + c < b + c$,

(b) $a > b \Rightarrow a + c > b + c$.

(iv) Multiplicative Property

If $a, b, c \in \mathbb{R}$ and $c > 0$, then

(a) $a > b \Rightarrow ac > bc$,

(b) $a < b \Rightarrow ac < bc$,

similarly, if $c < 0$ then,

(a) $a > b \Rightarrow ac < bc$

(b) $a < b \Rightarrow ac > bc$

(v) Reciprocative Property

If $a, b \in \mathbb{R}$ and a, b are of same sign then,

(a) If $a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$ and if $\frac{1}{a} < \frac{1}{b} \Rightarrow a > b$

(b) If $a > b \Rightarrow \frac{1}{a} < \frac{1}{b}$ and if $\frac{1}{a} > \frac{1}{b} \Rightarrow a < b$

(vi) Cancellation property

If $a, b, c \in \mathbb{R}$

(a) $a + c > b + c \Rightarrow a > b$

(b) $a + c < b + c \Rightarrow a < b$

similarly, (a) $ac > bc \Rightarrow a > b$, where $c > 0$

(b) $ac < bc \Rightarrow a < b$, where $c > 0$

Exercise 1.2

1. Recognize the properties of real numbers used in the following:

(i) $\frac{1}{2}+\frac{2}{3}=\frac{2}{3}+\frac{1}{2}$

(ii) $\frac{4}{3}+\left(1\frac{1}{3}+\frac{2}{3}\right)=\left(\frac{4}{3}+1\frac{1}{3}\right)+\frac{2}{3}$

(iii) $9\times\left(\frac{10}{9}+\frac{20}{9}\right)=\left(9\times\frac{10}{9}\right)+\left(9\times\frac{20}{9}\right)$

(iv) $\left(\frac{4}{5}+\frac{5}{7}\right)\times\frac{7}{8}=\left(\frac{4}{5}\times\frac{7}{8}\right)+\left(\frac{5}{7}\times\frac{7}{8}\right)$

(v) $\left(\frac{7}{5}-\frac{3}{5}\right)\times\frac{10}{15}=\left(\frac{7}{5}\times\frac{10}{15}\right)-\left(\frac{3}{5}\times\frac{10}{15}\right)$

(vi) $\frac{d}{c}\times\frac{e}{f}=\frac{e}{f}\times\frac{d}{c}$

(vii) $11\times(15\times21)=(11\times15)\times21$

(viii) $\frac{2}{11}\times\frac{11}{2}=\frac{11}{2}\times\frac{2}{11}=1$

(ix) $\left(\frac{3}{5}\right)+\left(-\frac{3}{5}\right)=\left(-\frac{3}{5}\right)+\left(\frac{3}{5}\right)=0$

(x) $\left(\frac{a}{b}\right)\times\left(\frac{b}{a}\right)=\left(\frac{b}{a}\right)\times\left(\frac{a}{b}\right)=1$

(xi) $\frac{15}{10}\times\left(\frac{8}{5}-\frac{4}{10}\right)=\left(\frac{15}{10}\times\frac{8}{5}\right)-\left(\frac{15}{10}\times\frac{4}{10}\right)$

(xii) $\frac{\sqrt{2}}{3}\times\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{2}}\times\frac{\sqrt{2}}{3}=1$

2. Fill the correct real number in the following to make the properties of real numbers correct.

(i) $\frac{\sqrt{2}}{5}+\frac{3}{\sqrt{6}}=\frac{\square}{\sqrt{6}}+\frac{\sqrt{2}}{5}$

(ii) $\frac{7}{10}+\left(\frac{70}{\square}+\frac{16}{33}\right)=\left(\frac{7}{\square}+\frac{\square}{10}\right)+\frac{16}{\square}$

(iii) $\frac{99}{50}\times\frac{50}{99}=\square$

(iv) $\left(\frac{59}{95}\right)\times\left(\frac{95}{59}\right)=\square$

(v) $(-21)+(\square)=0$

(vi) $\frac{5}{8}\times\left(\frac{2}{3}+\frac{5}{7}\right)=\left(\frac{\square}{\square}\times\frac{2}{3}\right)+\left(\frac{5}{8}\times\frac{\square}{\square}\right)$

3. Fill the following blanks to make the property correct/true.

(i) $5<8$ and $8<10\Rightarrow$ ____ $<$ ____

(ii) $10>8$ and $8>5\Rightarrow$ ____ $<$ ____

(iii) $3<6\Rightarrow3+9<$ ___ + ___

(iv) $4<6\Rightarrow4+8<$ ___ + ___

(v) $8>6\Rightarrow6+8>$ ___ + ___

4. Fill the following blanks which make the property correct/true:

(i) $5 < 7 \Rightarrow 5 \times 12 < \_\_\_ \times \_\_\_$

(ii) $7 > 5 \Rightarrow 7 \times 12 > \_\_\_ \times \_\_\_$

(iii) $6 > 4 \Rightarrow 6 \times (-7) \_\_\_\_ 4 \times (-7)$

(iv) $2 < 8 \Rightarrow 2 \times (-4) \_\_\_\_ 8 \times (-4)$

5. Find the additive and multiplicative inverse of the following real numbers.

(i) 3 **(ii)** -7 **(iii)** 0.3 **(iv)** $\frac{-\sqrt{5}}{5}$ **(v)** $\frac{9}{\sqrt{12}}$ **(vi)** 0

1.3 Radicals and Radicands.

1.3.1 Identify radicals and radicands

Let $n \in Z^+$ (Set of Positive integers) and $n > 1$,

also let $a \in \mathbb{R}$, then for any positive real number x,

such that $x^2 = a \Rightarrow x = a^{\frac{1}{2}} \Rightarrow x = \sqrt{a}$ (square root of a)

similarly, $x^3 = a \Rightarrow x = a^{\frac{1}{3}} \Rightarrow x = \sqrt[3]{a}$ (cube root of a)

$x^4 = a \Rightarrow x = a^{\frac{1}{4}} \Rightarrow x = \sqrt[4]{a}$ (4^{th} root of a)

In general, $x^n = a \Rightarrow x = a^{\frac{1}{n}} \Rightarrow x = \sqrt[n]{a}$ (n^{th} root of a)

In $\sqrt[n]{a}$, 'a' is called radicand and 'n' is called the index of the root.

The symbol $\sqrt{\ }$ is called radical sign.

1.3.2 Differentiate between Radical and Exponential forms of an Expression

As we have studied that $x = \sqrt[n]{a}$ is in a radical form.

Similarly, $a^{\frac{1}{3}}, a^{\frac{2}{3}}, a^{\frac{3}{2}}, a^{\frac{1}{n}}, a^{\frac{m}{n}}$ are some examples of exponential form.

Remember that

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

Here, $\sqrt[n]{a}$ is in radical form and $a^{\frac{1}{n}}$ is exponential form.

Here are some properties of square root

For all $a, b \in \mathbb{R}^+ \wedge m, n \in Z$

Then,

(i) $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

(ii) $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

(iii) $\dfrac{a}{\sqrt{a}} = \sqrt{a}$

(iv) $\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$

(v) $\dfrac{\sqrt{a}}{\sqrt{a}} = 1$

(vi) $m\sqrt{a} \pm n\sqrt{a} = (m \pm n)\sqrt{a}$

(vii) $\sqrt{\left(\dfrac{a}{b}\right)^{-n}} = \sqrt{\left(\dfrac{b}{a}\right)^{n}}$

Similarly,

(i) $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

(ii) $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

(iii) $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

(iv) $\sqrt[mn]{a^n} = a^{\frac{n}{nm}} = a^{\frac{1}{m}}$

(v) $\sqrt[mn]{a} = \sqrt[m]{a^{\frac{1}{n}}} = \sqrt[m]{\sqrt[n]{a}} = a^{\frac{1}{mn}}$

(vi) $\sqrt[n]{\sqrt[n]{a}} = \sqrt[n^2]{a} = a^{\frac{1}{n^2}}$

(vii) $\sqrt[n]{a^n} = a$

(viii) $\dfrac{\sqrt[n]{a^n}}{\sqrt[n]{a^n}} = 1$

1.3.3 Transform an Expression given in Radical Form to an Exponent Form and vice versa

The properties of radicals and exponential forms are very useful when we simplify the expressions involving radicals and exponents.

Example 01 Transform the following radical expressions into exponential forms.

(i) $\sqrt{\frac{2}{3}}$ (ii) $\sqrt[3]{18}$ (iii) $\sqrt[5]{\frac{5}{7}}$ (iv) $\sqrt[9]{\left(\frac{x}{y}\right)^2}$ (v) $\sqrt[4]{(ab)^3}$

Solutions:

(i) $\sqrt{\frac{2}{3}} = \left(\frac{2}{3}\right)^{\frac{1}{2}}$ (ii) $\sqrt[3]{18} = (18)^{\frac{1}{3}}$ (iii) $\sqrt[5]{\frac{5}{7}} = \left(\frac{5}{7}\right)^{\frac{1}{5}}$

(iv) $\sqrt[9]{\left(\frac{x}{y}\right)^2} = \left(\frac{x}{y}\right)^{\frac{2}{9}}$ (v) $\sqrt[4]{(ab)^3} = (ab)^{\frac{3}{4}}$.

Example 02 Transform the following exponential forms into radical expressions.

(i) $\left(\frac{5}{7}\right)^{\frac{1}{3}}$ (ii) $(12)^{\frac{n}{2}}$ (iii) $(-7)^{\frac{3}{4}}$ (iv) $\left(\frac{y}{x}\right)^{-\frac{2}{5}}$ (v) $\left(-\frac{x}{y}\right)^{\frac{m}{n}}$

Solutions:

(i) $\left(\frac{5}{7}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{5}{7}}$ (ii) $(12)^{\frac{n}{2}} = \sqrt{(12)^n}$ (iii) $(-7)^{\frac{3}{4}} = \sqrt[4]{(-7)^3}$

(iv) $\left(\frac{y}{x}\right)^{-\frac{2}{5}} = \sqrt[5]{\left(\frac{y}{x}\right)^{-2}} = \sqrt[5]{\left(\frac{x}{y}\right)^2}$ (v) $\left(-\frac{x}{y}\right)^{\frac{m}{n}} = \sqrt[n]{\left(-\frac{x}{y}\right)^m}$

Exercise 1.3

1. **Identify radicand and index in the following:**

(i) $\sqrt[3]{5}$ (ii) $\sqrt[4]{\frac{x}{y}}$ (iii) $\sqrt[5]{x^2yz}$ (iv) $\sqrt{ab}$ (v) $\sqrt[n]{\frac{pq}{r}}$

2. Transform the following into exponential forms.

(i) $\sqrt{\left(\frac{3}{4}\right)}$ (ii) $\sqrt{\left(\frac{x}{y}\right)^5}$ (iii) $\sqrt[3]{\left(\frac{y}{x}\right)^{-5}}$

(iv) $\sqrt[3]{(yz)^7}$ (v) $\sqrt[9]{27}$ (vi) $\sqrt[3]{(-64)^2}$

(vii) $\sqrt[3]{\left(\frac{1}{2}\right)^m}$ (viii) $\sqrt[5]{(xy)^3}$ (ix) $\sqrt[3]{\sqrt{\frac{4}{3}}}$

3. Transform the following into radical forms.

(i) $\left(5^3\right)^{\frac{1}{7}}$ (ii) $(ab^{-2})^{\frac{1}{3}}$ (iii) $\left[\left(\frac{5}{7}\right)^3\right]^{\frac{5}{7}}$

(iv) $\left(\frac{b}{a}\right)^{-\frac{m}{2}}$ (v) $\left[\left(\frac{11}{13}\right)\left(\frac{12}{13}\right)\right]^{\frac{1}{5}}$

1.4 Laws of Exponents/Indices:

Laws of exponents or indices are important in many fields of mathematics.

1.4.1 Recall Base, Exponent and value of Power

Consider an exponential form a^n here, **'*a*'** is called the base and **'*n*'** is called exponent i.e., read as a to the nth power. The result of a^n, where $a \in \mathbb{R}$ is called its value.

1.4.2 Apply the Laws of Exponents to Simplify Expressions with Real Exponents

The following laws of exponents are useful to simplify the expressions.

(i) Law of Product of Powers

(a) If $a, b \in \mathbb{R}$ and $x, y \in Z^+$

Then, $a^x \times a^y = a^{x+y}$

Some examples based on this law are given below:

(a) $a^2 \times a^3 = a^{2+3} = a^5$ (b) $3 \times 3^5 = 3^{1+5} \times 3^6 = 729$

(ii) Law of Power of Power

If $a \in \mathbb{R}$ and $x, y \in Z^+$, then $\left(a^x\right)^y = a^{xy}$

Some examples based on this law are given below:

(a) $(5^2)^4 = 5^{2\times 4} = 5^8$

(b) $\left\{\left(\frac{6}{11}\right)^4\right\}^3 = \left(\frac{6}{11}\right)^{4\times3} = \left(\frac{6}{11}\right)^{12}$

(c) $\left\{\left(-\frac{3}{4}\right)^3\right\}^3 = \left(-\frac{3}{4}\right)^{3\times3} = \left(-\frac{3}{4}\right)^9 = -\left(\frac{3}{4}\right)^9$

(iii) Law of Power of a Product

For all $a, b, \in \mathbb{R}$ and $n \in Z^+$,

Then, $(a\times b)^n = a^n \times b^n$

Following examples are based on this law:

(a) $(xy)^3 = x^3y^3$ (b) $\left\{\left[\frac{8}{9}\right]\left[\frac{7}{11}\right]\right\}^3 = \left(\frac{8}{9}\right)^3\left(\frac{7}{11}\right)^3$

(iv) Law of Power of a Quotient

For all $a, b, \in \mathbb{R}$ and $n \in Z^+$, then $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$

The following examples based on this law are given below:

(a) $\left(\frac{5}{8}\right)^3 = \frac{5^3}{8^3}$ (b) $\left(\frac{f}{g}\right)^4 = \frac{f^4}{g^4}$, $g\neq 0$

(v) Law of quotient of power

If $a \in \mathbb{R}$, $a \neq 0$ and $m, n \in Z^+$, then,

$$\frac{a^m}{a^n} = a^{m-n}, \text{ if } m > n$$

$$= \frac{1}{a^{n-m}}, \text{ if } n > m,$$

If $m = n$,

then, $a^{m-n} = a^{m-m} = a^0 = 1$

Similarly, $\frac{a^m}{a^n} = \frac{a^m}{a^m} = \frac{a^n}{a^n} = 1$

The following examples based on this law are given below:

(a) $\frac{3^5}{3^2} = 3^{5-2} = 3^3 = 27$

(b) $\frac{7^3}{7^5} = \frac{1}{7^{5-3}} = \frac{1}{7^2} = \frac{1}{49}$

Remember that:

$(-a)^n = a^n$, if n is an even exponent.
$= -a^n$, if n is an odd exponent.

Remember that:

If the exponent of a non-zero real number is zero then its value is equal to 1. For example: $3^0 = 1$.

Exercise 1.4

1. **Simplify the following:**

(i) $\dfrac{3^5}{3^2}$ (ii) $\dfrac{2^4 \cdot 5^3}{10^2}$ (iii) $\dfrac{(a+b)^2 \cdot (c+d)^3}{(a+b) \cdot (c+d)^2}$

2. **Simplify using law of exponent:**

(i) $\left(\dfrac{1}{3}\right)^4 \times \left(\dfrac{1}{3}\right)^5$ (ii) $\left(\dfrac{3}{4}\right)^5 \times \left(\dfrac{3}{4}\right)^2$ (iii) $\left(-\dfrac{4}{5}\right)^3 \times \left(-\dfrac{4}{5}\right)^5$

(iv) $(-3\times 5^2)^3$ (v) $[3\times(-4)^2]^3$ (vi) $\left(-\dfrac{a}{bc}\right)^5 \times \left(-\dfrac{a}{bc}\right)^4$

(vii) $\left(-\dfrac{c}{d}\right)^2 \left(-\dfrac{c}{d}\right)^3 \left(-\dfrac{c}{d}\right)^5$ (viii) $mn^2t^4n^3m^5t^7$ (ix) $a^2c^5b^2a^3c^3b^4a^4$

3. **Simplify using the law of exponent:**

(i) $(5^2)^3$ (ii) $\{(xy)^3\}^5$ (iii) $\{(-4)^2\}^5$

(iv) $\{(-3)^3(-4)^2\}^3$ (v) $\left(\dfrac{b^2}{5}\right)^3$ (vi) $\left\{\left(-\dfrac{4}{9}\right)^2\right\}^3$

(vii) $\{(z^3)^2\}^4$ (viii) $\{(mm^2m^3m^4)^2\}^5$ (ix) $-[(-0.1)^2\,(-0.1)^3\,(-0.1)^4]^2$

1.5 Complex Numbers

1.5.1 Elucidate, then define a complex number z represented by an expression of the form (*a*,*b*) or $z = a + ib$, where a is real part and b is imaginary part.

We know that the square of real number is non-negative. So the solution of the equation $x^2+1=0$ does not exist in $\mathbb{R}$. To overcome this inadequacy of real number, mathematicians introduced a new number $\sqrt{-1}$, imaginary unit and denoted it by the letter i (iota) having the property that $i^2=-1$. Obviously i is not real number. It is a new mathematical entity that enables us to find the solution of every algebraic equation of the type $x^2+a=0$ where $a>0$. Numbers like $\sqrt{-1}=i, \sqrt{-5}=\sqrt{5}i, \sqrt{-49}=7i$ are called pure imaginary number.

Definition of Complex Number

A number of the form $a+ib$ where a and b are real numbers and i is an imaginary unit i.e. $i=\sqrt{-1}$ is called a complex number and it is denoted by z. e.g. $z=3+4i$ is a complex number.

The complex number $a+ib$ can be written in ordered pair form (a,b) such as $5+8i=(5,8)$.

1.5.2 Recognize a as real part and b as imaginary part of $z=a+ib$

In the complex number $z=a+ib$, "a" is the real part of complex number and "b" is the imaginary part of complex number. The real part of complex number is denoted by Re(z) and its imaginary part is denoted by Im(z).

Example 01 Recognize real and imaginary parts for the given complex number.

$z=3-2i$

Here, Re(z)= $a=3$ and Im(z)= $b=-2$

1.5.3 Define conjugate of a complex number

Conjugate of z is denoted by $\bar{z}$ i.e.,

If $z=a+ib$ then $\bar{z}=a-ib$ or If $z=(a,b)$, then $\bar{z}=(a,-b)$

and, if $z=a-ib$, then $\bar{z}=a+ib$ or and If $z=(a,-b)$, then $\bar{z}=(a,b)$

- In conjugate of complex number we just change the sign of its imaginary part.

Note: If any complex number z then $\overline{(\bar{z})}=z$

Example 01 Find the conjugate of the following complex numbers.

(i) $3+4i$ (ii) $\left(-\frac{4}{5},\frac{5}{4}\right)$ (iii) $(-3,0)$

Solutions:

(i) Let $z_1=3+4i$

then $\bar{z}_1=\overline{3+4i}$

$\bar{z}_1=3-4i$

(ii) Let $z_2=\left(-\frac{4}{5},-\frac{5}{4}\right)$

$\bar{z}_2=\overline{\left(-\frac{4}{5},-\frac{5}{4}\right)}$

$\bar{z}_2=\left(-\frac{4}{5},\frac{5}{4}\right)$

(iii) Let $z_3=(-3,0)$

$\bar{z}_3=\overline{(-3,0)}$

$\bar{z}_3=(-3,0)$

Note: The real number is self conjugate i.e $\overline{(a,0)}=\bar{a}=a, \forall\, a\in\mathbb{R}$

1.5.4 Know the condition of equality of complex numbers

Two complex numbers are said to be equal if and only if they have same real and imaginary parts. i.e.

$\forall a,b,c,d \in \mathbb{R}$, such that $a + ib = c + id$, iff $a = c$ and $b = d$.

Example 01 If $4x + 3yi = 16 + 9i$, find x and y.

Solution: Given that

$4x + 3yi = 16 + 9i \Rightarrow 4x = 16$ and $3y = 9$,

$\Rightarrow \frac{4x}{4} = \frac{16}{4}$ and $\frac{3y}{3} = \frac{9}{3} \Rightarrow x = 4$ and $y = 3$.

Example 02 If $x^2 + iy^2 = 25 + i36$, find x and y.

Solution: Given that

$x^2 + y^2 i = 25 + 36i \Rightarrow x^2 = 25$ and $y^2 = 36$,

$x = \pm\sqrt{25}$ and $y = \pm\sqrt{36} \Rightarrow x = \pm 5$ and $y = \pm 6$

Exercise 1.5

1. **Write the following complex numbers in the form of $a+ib$.**

(i) (1,2) (ii) (2,2) (iii) (0, 4)

(iv) (–1,1) (v) (–2, 0) (vi) (–3,4)

2. **Identify real and imaginary parts for the following complex numbers.**

(i) $1+2i$ (ii) $9i+4$ (iii) $(-5,6)$

(iv) $-1-i$ (v) $\left(-\frac{3}{4}\right)-\left(-\frac{4}{5}\right)i$ (vi) $2i-1$

3. **Find the conjugate of the following complex numbers.**

(i) $3+2i$ (ii) $(0, -7)$ (iii) $(-1, 0)$

(iv) $1-i$ (v) $\left(-\frac{3}{4}\right)+\left(-\frac{4}{5}\right)i$ (vi) $3i+1$

4. **Verify that $\overline{(\overline{z})} = z$, for the following complex numbers.**

(i) $\left(\frac{4}{7}\right)+\left(\frac{9}{10}\right)i$ (ii) $\left(-\frac{9}{11}\right)+\left(\frac{10}{9}\right)i$ (iii) $\frac{1}{2}-3i$

(iv) $2+3i$ (v) $-2-3\left(-\frac{10}{9}\right)i$ (vi) $4x+3iy$

5. **Find the values of x and y, when**

(i) $x+yi=-5+5i$ (ii) $x^2+iy^2=\frac{16}{9}+\frac{9}{25}i$

(iii) $y^2+\frac{x}{3}i=121-\frac{9}{5}i$ (iv) $\frac{\sqrt{5}}{3}x-\frac{3}{\sqrt{2}}yi=\frac{6\sqrt{3}}{\sqrt{2}}+\frac{2\sqrt{2}}{9}i$

1.6 Basic Operations on Complex Numbers

1.6.1 Carry out Basic Operations (Addition, Subtraction, Multiplication and Division) on Complex Numbers

(i) Addition of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers $\forall a, b, c, d \in \mathbb{R}$, then their sum,

$z_1 + z_2 = (a + ib) + (c + id)$

$= (a + c) + i(b + d) = (a + c, b + d)$.

Remember that:

$(a,b)+(c,d)=(a+c,b+d)$

Example: If $z_1 = 6 + 9i$ and $z_2 = -1 + 2i$, find $z_1 + z_2$.

Solution: Given that $z_1 = 6 + 9i = (6,9)$ and $z_2 = -1 + 2i = (-1,2)$

we know that $z_1 + z_2 = (a + c) + i(b+d) = (a + c, b + d)$

$\therefore$ $z_1 + z_2 = (6, 9) + (-1, 2)$

$\Rightarrow$ $z_1 + z_2 = (6 - 1, 9 + 2)$

$\Rightarrow$ $z_1 + z_2 = (5, 11)$

(ii) Subtraction of complex numbers.

Let $z_1 = a + ib$ and $z_2 = c + id$, $\forall a, b, c, d \in \mathbb{R}$,

then $z_1 - z_2 = (a + ib) - (c + id)$

$= (a - c) + i.(b - d) = (a - c, b - d)$

Example: If $z_1 = -7 + 2i$ and $z_2 = 4 - 9i$, find $z_1 - z_2$.

Solution: Given that $z_1 = -7 + 2i = (-7,2)$ and $z_2 = 4 - 9i = (4,-9)$

we know that $z_1 - z_2 = (a - c, b - d)$

$\therefore$ $z_1 - z_2 = (-7 - 4, 2+9)$

$\Rightarrow$ $z_1 - z_2 = (-11, 11)$

Remember that:

$(a,b)-(c,d)=(a-c,b-d)$

(iii) Multiplication of complex numbers.

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, $\forall a,b,c,d \in \mathbb{R}$

$$z_1 \cdot z_2 = (a+ib)(c+id)$$
$$= c(a+ib)+di(a+ib)$$
$$= ac+bci+adi+bdi^2$$
$$=(ac-bd)+i(ad+bc)=(ac-bd,ad+bc) \qquad i^2=-1$$

Remember that:

$(a,b)(c,d)=(ac-bd,ad+bc)$

Example If $z_1 = 3+4i=(3,4)$ and $z_2=-3-4i=(-3,-4)$, find the product z_1z_2.

Solution: Given that

$z_1 = 3+4i=(3,4)$ and $z_2=-3-4i=(-3,-4)$

We know that $z_1z_2=(ac-bd,ad+bd)$

$\therefore \quad z_1z_2=(3,4)\cdot(-3,-4)$

$\Rightarrow \quad z_1z_2=(-9+16,-12-12)=(7,-24)$

(iv) Division of complex numbers

Let $z_1=a+ib=(a,b)$ and $z_2=c+id=(c,d)$, $z_2\neq0$.

Division of complex number z_1 by another complex number z_2 written as under

$$\frac{z_1}{z_2}=\frac{a+ib}{c+id}$$
$$=\frac{a+ib}{c+id}\times\frac{c-id}{c-id}$$
$$=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$=\left(\frac{ac+bd}{c^2+d^2}\right)+i\left(\frac{bc-ad}{c^2+d^2}\right)$$
$$=\left(\frac{ac+bd}{c^2+d^2},\frac{bc-ad}{c^2+d^2}\right)$$

Remember that:

$$\frac{(a,b)}{(c,d)}=\left(\frac{ac+bd}{c^2+d^2},\frac{bc-ad}{c^2+d^2}\right)$$

Example 01 Simplify: $\dfrac{2+3i}{4+2i}$

Solution: $\dfrac{2+3i}{4+2i}$

Multiplying and dividing by conjugate of denominater, we have

$$=\frac{2+3i}{4+2i}\times\frac{4-2i}{4-2i}$$
$$=\frac{(8+6)+i(12-4)}{(4)^2-(i2)^2}$$
$$=\frac{14+8i}{20}$$
$$=\frac{14}{20}+i\frac{8}{20}$$
$$=\frac{7}{10}+i\frac{4}{10}$$
$$=\left(\frac{7}{10},\frac{4}{10}\right)=\left(\frac{7}{10},\frac{2}{5}\right)$$ Hence simplified.

Example 02 Perform division of complex numbers using division formula.
$(-1,3)\div(2,-4)$.

Solution: **Formula:** $\dfrac{z_1}{z_2}=\dfrac{(a,b)}{(c,d)}=\left(\dfrac{ac+bd}{c^2+d^2},\dfrac{bc-ad}{c^2+d^2}\right)$

$$\frac{(-1,3)}{(2,-4)}=\left(\frac{(-1)(2)+(3)(-4)}{2^2+(-4)^2},\frac{(3)(2)-(-1)(-4)}{2^2+(-4)^2}\right)$$
$$=\left(\frac{-2-12}{4+16},\frac{6-4}{4+16}\right)$$
$$=\left(\frac{-14}{20},\frac{2}{20}\right)$$
$$=\left(\frac{-7}{10},\frac{1}{10}\right)$$

Exercise 1.6

1. **Perform the indicated operations of the following complex numbers.**

(i) $(3,2)+(9,3)$ (ii) $\left(\frac{3}{2},\frac{2}{3}\right)+\left(\frac{2}{3},\frac{3}{2}\right)$

(iii) $(15,12)-(10,-9)$ (iv) $\left(\frac{4}{5},\frac{8}{15}\right)-\left(\frac{4}{5},\frac{6}{10}\right)$

(v) $(1,2)(1,-2)$ (vi) $(4,-5)(5,-4)$

(vii) $(3,-7)\div(3,2)$ (viii) $(4,5)\div(2,-3)$

2. **Simplify and write your answer in form of $a+ib$**

(i) $\frac{-1}{1+i}$ (ii) $(1+i)^4$ (iii) $\left(\frac{1}{1+i}\right)^2$ (iv) $(1+i)^8$

3. **If $z_1=-4+6i$ and $z_2=2\frac{1}{2}-2i$, verify that**

(i) $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$ (ii) $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$

4. **If $z_1=1+i$ and $z_2=1-i$, verify that**

(i) $\overline{z_1 z_2}=\overline{z_1}\cdot\overline{z_2}$ (ii) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z_1}}{\overline{z_2}}$

Review Exercise 1

1. **Fill in the blanks.**

(i) Multiplicative inverse of $\sqrt{5}$ is ___________.

(ii) $Q\cup Q'=$ ___________.

(iii) The additive identity in $\mathbb{R}$ is ___________.

(iv) 5+(6+7)=(5+6)+ ___________.

(v) 3+(−3)= ___________.

(vi) π is a ___________number.

(vii) $\frac{22}{7}$ is a ____________number.

(viii) The conjugate of $-3 + 5i$ is ____________.

(ix) In $2i(3-i)$, the real part is ____________.

(x) The product of two complex numbers (a,b) and (c,d) i.e. $(a,b).(c.d)$ = ____________.

2. Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) $\mathbb{R}$ is closed under multiplication. T / F

(ii) if $x < y \wedge y < z \Rightarrow x < z$. T / F

(iii) $\forall x, y, z \in \mathbb{R}, x(y-z) = xy - xz$ T / F

(iv) The product of every two imaginary numbers is real. T / F

(v) The sum of two real numbers is a real number. T / F

3. Tick (✓) the correct answer.

(i) The additive inverse of $\sqrt{5}$ is

(a) $-\sqrt{5}$ (b) $\frac{1}{\sqrt{5}}$ (c) $\sqrt{-5}$ (d) -5

(ii) $(5i).(-2i) =$

(a) -10 (b) 10 (c) $-10i$ (d) $10i$

(iii) 3(5+7)=3.5+3.7, name the property used

(a) Commutative (b) Associative (c) Distributive (d) Closure

(iv) $\sqrt{-2} \times \sqrt{-2} =$

(a) 2 (b) -2 (c) $2i$ (d) $-2i$

4. Simplify (i) $2^{3^2} \div 2^{2^3}$ (ii) $3^{2^3} \div 3^{3^2}$

5. Simplify (i) $3^{20} + 3^{20} + 3^{20}$ (ii) $2^{35} + 2^{35}$

6. Let z = 7-i, find

(i) Re $(\bar{z})$ (ii) Im (z) (iii) $\bar{z}$ (iv) $|\bar{z}|$

(v) z^{-1} (vi) $|z|^{-1}$ (vii) iz (viii) $i\bar{z}$

Summary

- The set of real numbers is the union of set of rational and irrational numbers i.e., $\mathbb{R} = Q \cup Q'$.
- There are two types of non–terminating decimal fractions i.e., non–recurring decimal fractions and recurring decimal fractions.

- Properties of real numbers w.r.t. "+" and "×"
  - (i) **Closure properties:**
    $a+b\in\mathbb{R}$ and $ab\in\mathbb{R}, \forall a,b\in\mathbb{R}$
  - (ii) **Associative properties:**
    $a+(b+c)=(a+b)+c$ and $a(bc)=(ab)c, \forall a,b,c\in\mathbb{R}$
  - (iii) **Commutative properties:**
    $a+b=b+a$ and $ab=ba, \forall a,b\in\mathbb{R}$
  - (iv) **Identity properties:**
    $a+0=a=0+a$ and $a\cdot 1=a=1\cdot a \;\forall a\in\mathbb{R}$
    i.e 0 and 1 are respectively additive and multiplicative identities.
  - (v) **Inverses properties:**
    $a+(-a)=0=-a+a$ and $a\times\frac{1}{a}=1=\frac{1}{a}\times 1, \forall a,\in\mathbb{R}$ and $a\neq 0$
  - (vi) **Distributive property:**
    $a(b+c)=ab+ac$ or $(b+c)a=ba+ca, \forall a,b,c\in\mathbb{R}$
- **Number line:** A line used for representing real number is called a number line.
- **Radical, Radicand and Index of the Root:**
  In $\sqrt[n]{a}$
  - $\sqrt{\ }$ is called a radical sign.
  - a is called the radicand.
  - n is called the index of the root.
- **Laws of Exponent:**
  - (i) If $a,b\in\mathbb{R}$ and $x,y\in Z^+$, then $a^x\times a^y=a^{x+y}$.
  - (ii) If $a\in\mathbb{R}$ and $x,y\in Z^+$, then $(a^x)^y=a^{xy}$
  - (iii) $\forall a,b\in\mathbb{R}$ and $n\in Z^+$, then $(a\times b)^n=a^n\times b^n$
  - (iv) $\forall a,b\in\mathbb{R}$ and $n\in Z^+$, then $\left(\frac{a}{b}\right)^n=\frac{a^n}{b^n}$, provided $b\neq 0$.
- **Complex Number:** $z=a+ib=(a,b)$ is called complex number, where 'a' is real part and 'b' is an imaginary part of z and $i=\sqrt{-1}$.
- **Operation on two complex numbers** $z_1=a+ib$ **and** $z_2=c+id$

$$z_1+z_2=(a+c)+i(b+d)$$
$$z_1-z_2=(a-c)+i(b-d)$$
$$z_1z_2=(ac-bd)+i(ad+bc)$$
$$\frac{z_1}{z}=\left(\frac{ac+bd}{c^2+d^2}\right)+i\left(\frac{bc-ad}{c^2+d^2}\right),\; z_2\neq 0$$

Unit 2

• Weightage = 5%

LOGARITHMS

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Express a number in standard form of scientific notation and vice versa.
- Define logarithm of a real number to a base a as a power to which a must be raised to give the number (i.e., $a^x = y \Leftrightarrow \log_a y = x$, $a > 0$, $y > 0$ and $a \neq 1$).
- Define a common logarithm, characteristic and mantissa of log of a number.
- Use tables to find the log of a number.
- Give concept of antilog and use tables to find the antilog of a number.
- Use of calculator to find the log and antilog of a number.
- Differentiate between common and natural logarithms.
- Write, $\log_{10} y = \log y$ or simply $\log y$ and $\log_e(y)$ as $\ln y$,

  (i) $\log_{10} y = x \Leftrightarrow y = 10^x$,

  (ii) $\ln y = x \Leftrightarrow y = e^x$.
- Prove the following laws of logarithms.

  (i) $\log_a(mn) = \log_a m + \log_a n$

  (ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$,

  (iii) $\log_a m^n = n\log_a m$,

  (iv) $\log_a m . \log_m n = \log_a n$.
- Apply these laws of logarithms to convert lengthy processes of multiplication, division and exponentiation into easier processes of addition and subtraction etc.

Introduction :

Logarithms were introduced by the great muslim mathematician **Abu Muhammad Musa Al-Khawarizmi**. Letter on in the seventeen century **John Napier** developed the concept of logarithm further and prepared tables for it. In these table the base *"e"* was used. ***e*** is an irrational number whose approximate value is 2.71828… . In 1631, Professor **Henry Briggs** developed the tables with base "10".

By the use of logarithms the enormous labour of calculations is reduced and it is performed with great ease.

2.1 Scientific Notation

Scientific notation is special form to write very large or very small numbers conveniently.

2.1.1 Express a number in standard form of scientific notation and vice versa

In the world of science and technology; we deal with very large and small numbers and quantities, the distance from the earth to the sun is **150,000,000 km** approximately and weight of hydrogen atom is **0.000,000,000,000,000,000,000,000,001,7g**.The writing of such type of numbers in **ordinary notation**(Standard notation) is too difficult for everyone and it is time consuming. Scientists have developed a convenient method to write very small and very large numbers that is called **scientific notation.**

The above mentioned number in section 2.1.1 can be simply written in scientific notation as: 1.5×10^{8} km and 1.7×10^{-27}g respectively.

The following examples will help to understand the scientific notation.

Example 01
Express the following numbers in scientific notation.

(i) 400900 (ii) 0.0000075

Solution:

(i) 400900

In given number, decimal is after the unit digit, so move decimal point up to five digits from right to left, and write as $400900 = 4.009 \times 10^5$, which is the required scientific notation.

(ii) 0.0000075

There are 7 digits after decimal point in the given decimal number. There is '7' first non-zero digit in it, so, we move decimal point up to 6 digits from left to right and write as $0.0000075 = 7.5 \times 10^{-6}$, which is the required scientific notation of the given number.

Example 02 Write the following in ordinary notation

(i) 2.76×10^6 (ii) 5.24×10^{-4}

Solution:

(i) 2.76×10^6

The power of 10 is 6, so we move decimal point from left to right up to six decimal places but there are 2 digits, so, we put 4 zeros from right side then the required ordinary form is $2.76 \times 10^6 = 2760000$, is the required ordinary notation.

Solution:

(ii) 5.24×10^{-4}

There is negative 4^{th} power of 10, so we move decimal point from right to left up to 4 decimal places but there is already one digit before decimal point, so, we put three zeros before the digit 5, and then we get the required notation.

Thus, $5.24 \times 10^{-4} = 0.000524$ is the required notation.

Exercise 2.1

1. Express each of the following numbers in scientific notation.

(i)	9700	(ii)	4,980,000	(iii)	96,000,000
(iv)	4169	(v)	84,000	(vi)	0.718
(vii)	0.00643	(viii)	0.0074	(ix)	0.21005

2. Express the following numbers in ordinary notation (Standard notation).

(i)	7×10^4	(ii)	8.072×10^{-10}	(iii)	6.018×10^6
(iv)	7.865×10^8	(v)	2.05×10^{-4}	(vi)	7.25×10^{10}
(vii)	4.502×10^6	(viii)	2.865×10^{-8}	(ix)	3.056×10^6

2.2 Logarithms

Logarithms is a method of reducing complicated problems of multiplication/ division / exponents into simple form.

2.2.1 Define logarithm of a real number to abase a as a power to which a must be raised to give the number (i.e. $a^x = y \Leftrightarrow \log_a y = x$, $a>0$, $y>0$ and $a \neq 1$)

If $a^x = y$, then x is called the logarithm of y to the base 'a' and is written as $\log_a y = x$, where, $a>0$, $y>0$ and $a \neq 1$.

Thus, $a^x = y \Leftrightarrow \log_a y = x$.

It is noted that $a^x = y$ is an exponential form and $\log_a y = x$ is a logarithmic form. Both the forms are interconvertible.

The following examples will help to understand the concept of exponential and logarithmic forms.

Example 01 Write $2^{-4} = \frac{1}{16}$ in logarithmic form.

Solution: $2^{-4} = \frac{1}{16} \Rightarrow \log_2 \frac{1}{16} = -4$

Example 02 Write $\log_3 81 = 4$ in exponential form.

Solution: $\log_3 81 = 4 \Rightarrow 3^4 = 81$

Example 03 Find the value of $\log_4 2$.

Solution: Let $x = \log_4 2$

Exponential form is

$\therefore \quad 4^x = 2$

$\Rightarrow \quad (2)^{2x} = 2^1$

Equating exponents on both the sides, we have

$2x = 1$

$\Rightarrow \quad x = \frac{1}{2}$

Example 04 Find the value of x if $\log_x 8 = \frac{3}{2}$

Solution: $\log_x 8 = \frac{3}{2}$

Exponential form is

$\Rightarrow \quad (x)^{\frac{3}{2}} = 8$

$\Rightarrow \quad (x)^{\frac{3}{2}} = 2^3$

Taking power $\frac{2}{3}$ on both sides, we have

$\Rightarrow \quad (x^{\frac{3}{2}})^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$

$\Rightarrow \quad x = 2^2$

$\Rightarrow \quad x = 4$

Example 05 Find the value of x if $\log_{64} x = \frac{-2}{3}$

Solution:

Exponential form is

$(64)^{\frac{-2}{3}} = x \quad \Rightarrow (4^3)^{\frac{-2}{3}} = x$

$4^{-2} = x \quad \Rightarrow \quad \frac{1}{4^2} = x$

$\frac{1}{16} = x \quad \Rightarrow \quad x = \frac{1}{16}$

Example 06 Find the value of x if $\log_5 5 = x$

Solution:

Converting $\log_5 5 = x$ into exponential form, we have

$5^x = 5$

Equating the exponents, we get

$\boxed{x = 1}$

Note: The logarithm of any positive number to itself is always 1.

Exercise 2.2

1. Write the following in logarithmic form.

(i) $7^3 = 343$ (ii) $3^{-4} = \frac{1}{81}$

(iii) $10^{-3} = 0.001$ (iv) $\sqrt[3]{8^2} = 4$

2. Write the following in exponential form.

(i) $\log_{27} 81 = \frac{4}{3}$ (ii) $\log_2 \frac{1}{8} = -3$

(iii) $\log_{10} 1 = 0$ (iv) $\log_{10}(0.01) = -2$

3. Find the value of unknown in the following.

(i) $\log_{32} x = \frac{1}{2}$ (ii) $\log_a 3 = \frac{1}{2}$ (iii) $\log_{\sqrt{5}} 25 = y$

(iv) $\log_4 x = \frac{3}{2}$ (v) $\log_{10} 100 = y$ (vi) $\log_a 64 = 3$

(vii) $\log_a 1 = 0$ (viii) $\log_{55} 55 = y$ (ix) $\log_{64} 8 = \frac{x}{2}$

2.2.2 Define a common logarithm, characteristic and mantissa of log of a number

Common logarithms

Common logarithms have base 10, it is also named as artificial logarithms or Briggs logarithm.
Common log written as $\log_{10} y$ or simply log$y$.

$$\text{If } \log y = x \Leftrightarrow y = 10^x$$

Characteristic and Mantissa of log of a number

Logarithms of a number consist of two parts. One part is integer and the second part is decimal fraction. Integral part is called Characteristic and decimal part is called Mantissa.

It is noted that characteristic of logarithm may be positive or negative, but mantissa is always positive, for this we use logarithmic tables.

In scientific notation, the power of 10 is called characteristic and mantissa is found by using log table which will be discussed later.

Example 01 Find the characteristic of the following numbers.

0.765, 0.04, 0.004567, 2.134, 23.56 and 3456.

Nos.	Number	Scientific Notation	Characteristic
1	0.765	7.65×10^{-1}	-1 or $\overline{1}$
2	0.04	4.0×10^{-2}	-2 or $\overline{2}$
3	0.0045467	4.5467×10^{-3}	-3 or $\overline{3}$
4	2.134	2.134×10^{0}	0
5	23.56	2.356×10^{1}	1
6	3456	3.456×10^{3}	3

We observe that

- Characteristic of logarithm of a number greater than 1 is always non-negative integer.
- Characteristic of logarithm of a number less than 1 is always negative.

Mantissa: The mantissa is found by using logarithmic tables. These tables are constructed to obtain the logarithms up to 7-decimal digits. But at this level, for practical purposes, a **four figure logarithmic table** is useful for accuracy to find the logarithm of a number.

Do you understand?
Negative characteristic of logarithm is written as: $\overline{3}, \overline{2}$ or $\overline{1}$ instead of $-3, -2$ or -1 respectively.
When mantissa becomes negative, then, we must change it into +ve number, because mantissa is always positive.

2.2.3 Use table to find the log of a number.

The following will help us to find the logarithm by using table.

Example 01 Find the Mantissa of the following logarithmic numbers

(i) log (43.254) (ii) log (0.002347).

Solution (i): log (43.254)

Step 1: Ignore decimal and round off the number up to 4 digits. Then we have number is 4325.

Step 2: Locate the row corresponding to 43 in log table.

Step 3: Proceed horizontally to third digit **i.e.** 2. The number at the intersection of 43^{rd} row and 2^{nd} column is 6355.

Step 4: Again, proceed horizontally till mean difference column till 4^{th} digit i.e.5, we get number 5 at the intersection of 5^{th} column and 43^{rd} row.

Step 5: Add 5 in 6355; we will get 0.6360 as the mantissa of log (43.25).

Logarithm Table

											Mean Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0017	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	2560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	6	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7

Logarithm Table

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	3	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	882	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Solution (ii):

For log(0.002347) we ignore decimal and zeros before the digit 2 and see in the log table at the intersection of row 23 and column 4 is 3692. Add mean difference column corresponding to the digit 7 is 13 in 3692, we get 3705. The required Mantissa is 0.3705.

So, Mantissa of log (0.002347) is 0.3705.

Example 02 Find the log of the following numbers:

(i) 278.27 (ii) 0.07058

Solution: Let $x = 278.27$

Taking log both sides,

$\therefore$ Log$x$ =log (278.27),

Step 1: Round off the number up to 4 Digits i.e. 278.3.

Step 2: $278.3 = 2.783 \times 10^2$

So, characteristics is =2.

Step 3: For finding mantissa ignore decimal point, we get 2783.

By using log table, we get mantissa of log (2783) = 0.4445.

Step 4: Add characteristic and mantissa.

We get, log $x = 2.4445$.

Solution(ii): Let $x = 0.07058$

Step 1: No need to round off here. Four digits are 7058.

Step 2: Convert the given number into scientific notation

i.e., 7058×10^{-2}.

so, characteristic = -2or $\bar{2}$.

Step 3: Ignore decimal point and find mantissa of 7058.

By using log table, mantissa of 7058 is 0.8487.

Step 4: Add characteristic and mantissa.

We get, Log x =log (0.07058) =$\bar{2}$.8487.

Remember that:

The logarithms of numbers of the same sequence of significant digits have the same mantissa.

For example, the numbers 0.004576, 0.04576, 0.4576, 45.76 etc. have the same mantissa.

Exercise 2.3

1. **Find the characteristics and mantissa of the following Logarithm.**

(i)	8	(ii)	5054	(iii)	9.992
(iv)	765.3	(v)	0.00329	(vi)	0.0000300

2. **Find the logarithms of the following numbers.**

(i)	9	(ii)	55.56	(iii)	29.592
(vi)	405.3	(v)	0.00469	(vi)	0.000076

3. **If log 31.09=1.4926, find the value of the following without using log table.**

(i)	log 3.109	(ii)	log 310.9	(iii)	log 0.003109
(iv)	log 3109	(v)	log 310.942	(vi)	log 310926

2.2.4 Give concept of antilog and use of tables to find the antilog of a number.

If log $x = y$, then x is called anti log of y. It is written as x = antilog y. If the common logarithm of a number x is y, i.e. if log $x = y$, then we find the number x by using the tables of antilogarithms and with the help of following two rules.

Rule 1. If the characteristic is non negative n, then antilog must have n+1 digits in integral part.

Rule 2. If the characteristic is negative n, then antilog must have n–1 zeros immediately following the decimal point.

The procedure of finding antilogarithms is explained by the following examples

Antilogarithm Table

											Mean Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	1002	1005	1007	1009	1012	1014	1016	1019	1021	0000	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	1	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2

Antilogarithm Table

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
.26	1820	1821	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6

Antilogarithm Table

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2970	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	2	4	4	5	6	7
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4374	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	5012	5023	5035	5047	5058	4070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	6310	6324	6339	6353	6368	6383	6397	6415	6427	6442	1	3	4	6	7	9	10	12	13
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14

Antilogarithm Table

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	7079	7096	7112	7129	7145	7161	7178	7196	7211	7228	2	3	5	7	8	10	12	13	15
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8275	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	####	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Example 01 Find the number whose logarithm is,

(i) 1.3247 (ii) $\bar{2}.1324$

Solution (i): Let x = antilog (1.3247), Here log x = 1.3247

Step 1: Now characteristic = 1 and mantissa = 0.3247

Step 2: Now locate the row corresponding to .32 in the antilog table

Step 3: Proceed horizontally to third digit that is 4. The number at the intersection of row 32^{nd} and column 4^{th} is 2109.

Step 4: Again, proceed horizontally go to mean difference 7^{th} column where the value is 3.

Step 5: Add 3 in 2109, we get 2112 as the required digits.

Step 6: Since characteristic is 1, so put decimal after two places from left to right, thus require antilog is 21.12.

Solution (ii): Let x = antilog ($\bar{2}.1324$), Here Log $\bar{x}$ = $\bar{2}.1324$

Here, Characteristic = $\bar{2}$ and mantissa = 0.1324

Now see .13 in anti-log table corresponding to column 2, we found 1355 and mean difference in 4^{th} column is 1, so it is 1355+1=1356.

Characteristics is -2,

thus required number is 0.01356.

2.2.5 Use of calculator to find the log and antilog of a number

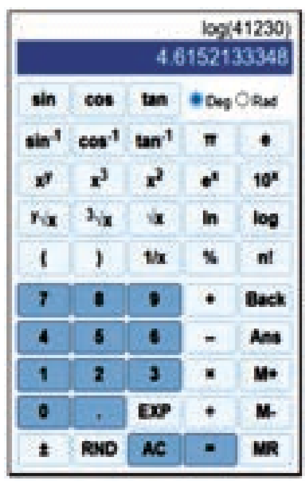
Example 1. By using calculator, determine the value of log (41230).

Solution: Let x = log (41230). Our first step is to press the 'Log' key.

Now enter (41230), (We want to determine its value.)

Finally, close the parenthesis and press the "=" key.

Now, we can see the value of the log (41230) on the screen which is, 4.615213335.



Thus, log (41230) = 4.615213335

Example 02: By using calculator, determine the value ofanti-log (4.615213335).

Solution: We have to use the antilog function key.

(i) Press 2nd function key or shift key.

(ii) Press the 'Log' key

(iii) Enter 4.615213335 followed by the right parenthesis symbol

(iv) Press 'ENTER' key

The answer of antilog 4.615213335 is 41230.00002 This number is rounded off to 41230.

Thus, antilog (4.615213335) = 41230.00002

2.3 Differentiate between common and natural logarithm.

The common logarithm has base 10, and is represented as log(x) instead of $\log_{10}(x)$, while natural logarithm has base **e** (e is an irrational number whose value is 2.718281…) and is represented as *lnx* instead of $\log_e(x)$.

Exercise 2.4

1. By using table, find the numbers whose common logarithms are.

(i) 3.56721 (ii) $\bar{1}.7427$ (iii) 0.35749

(iv) 5.8196 (v) $\bar{4}.3847$ (vi) 0.9187

2. Find the Logarithm of the following numbers by using calculator.

(i) 900 (ii) 45.54 (iii) 36582

(iv) 826.3 (v) 0.00851 (vi) 0.000097

3. Find the value of x from the following, using calculator.

(i) $\log x = 1.7505$ (ii) $\log x = 0.6609$ (iii) $\log x = \bar{1}.6132$

(iv) $\log x = 3.4800$ (v) $\log x = \bar{7}.0038$ (vi) $\log x = 0.2665$

2.4 Laws of Logarithms.

2.4.1 Prove the following laws of logarithms.

(i) $\log_a(mn) = \log_a m + \log_a n$

(ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n \log_a m$

(iv) $\log_a n = \dfrac{\log_b n}{\log_b a}$

(i) For real numbers m, n, a and $a > 0, \ a \neq 1, \log_a(mn) = \log_a m + \log_a n$

Proof: Let $\log_a m = x$ and $\log_a n = y$. Then

$\Rightarrow \quad m = a^x$ and $n = a^y$

Now $mn = a^x \cdot a^y$

$mn = a^{x+y}$ (Rule of indices)

By changing exponential form into logarithmic form

$\log_a (mn) = x+y$

Hence, $\log_a (mn) = \log_a m + \log_a n$

The logarithm of the product of two numbers is the sum of their logarithms.

(ii) For real numbers m, n, a and $a > 0, a \neq 1,$

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

Proof: Let $\log_a m = x$ and $\log_a n = y$. Then

$m = a^x$ and $n = a^y$

Now $\dfrac{m}{n} = \dfrac{a^x}{a^y}$

$\dfrac{m}{n} = a^{x-y}$

By changing exponential form into logarithmic form

$\Rightarrow \quad \log_a \dfrac{m}{n} = x - y$

Hence, $\log_a \dfrac{m}{n} = \log_a m - \log_a n$

The logarithm of the quotient of two numbers is the difference of their logarithms.

(iii) For real numbers m, n, a and $a > 0, \ a \neq 1,$

$$\log_a m^n = n \log_a m$$

Proof: Let $\log_a m = x$ Then $m = a^x$

Now $m^n = (a^x)^m$

$m^n = a^{nx}$

By changing exponential form into logarithmic form

$\Rightarrow \quad \log_a m^n = nx$

Hence $\log_a m^n = n\log_a m$

The logarithm of a number raised to a power n is the product of the exponent n and the logarithm of the number.

(iv) Change of base property

For real numbers a, b, n *and* $a>0, \ a \neq 1,$

$$\log_a n = \frac{\log_b n}{\log_b a}$$

Proof: Let $\log_a n = x$

So, $n = a^x$

Taking logarithms of both sides to the base b, we have

$\log_b n = \log_b a^x$

$\log_b n = x\log_b a \qquad \therefore \ \log_b m^n = n\log_b m$

$$x = \frac{\log_b n}{\log_b a}$$

Hence $\log_a n = \dfrac{\log_b n}{\log_b a}$

Example 01 Express $\log_a(2bc)$ as a sum of logarithms.

Solution: Using the Law of logarithm,

$\log_a(2bc) = \log_a 2 + \log_a b + \log_a c,$

Hence expressed as sum of the logarithms.

Example 02 Express $\log(52.5 \times 63 \times 4.567)$ as a sum of the logarithms.

Solution: Using the Law of logarithm,

$\log(52.5 \times 63 \times 4.567) = \log 52.5 + \log 63 + \log 4.567$

Hence expressed as some of the logarithms.

Note that

(i) $\log_a(mn) \neq \log_a m \times \log_a n$

(ii) $\log_a m + \log_a n \neq \log_a (m + n)$

Example 03 Express $\log\left(\frac{213.1}{34.22}\right)$ as a difference of logarithms

Solution: Apply difference law of log, on $\log\left(\frac{213.1}{34.22}\right)$, we have,

$$\log\left(\frac{213.1}{34.22}\right) = \log 213.1 - \log 34.22.$$

Hence expressed as a difference of logarithms.

Example 04 Express $\log_a 2^x$ as a product.

Solution: We know that $\log_a m^n = n\log_a m$

$\therefore$ $\log_a 2^x = x \log_a 2.$

Exercise 2.5

1. Express the following logarithm in terms of $\log_a x$, $\log_a y$ and $\log_a z$.

(i) $\log_a (xyz)$ (ii) $\log_a (x^2y)$ (iii) $\log_a\left(\frac{xy}{z}\right)$

(iv) $\log_a \sqrt{xy}$ (v) $\log_a\left(\frac{1}{\sqrt{xyz}}\right)$ (vi) $\log_a \frac{x^3y}{z^2}$

(vii) $\log_a \sqrt{xy^2z}$ (viii) $\log_a\left(\sqrt[3]{x^{-1}\sqrt{y^3}} \div \sqrt{y^3\sqrt{x}}\right)$

(xi) $\log_a \frac{x\sqrt{y^3}}{\sqrt[3]{z^2x^5}}$

2. Reduce each of the following into a single term.

(i) $\log_a 20 - \log_a 15 + \frac{1}{2}\log_a \frac{9}{2}$

(ii) $\frac{1}{3}\log_a(x-1)^3 + \frac{10}{9}\log_a(x+1) - \frac{1}{9}\log_a(x+1)$

(iii) $\log x - 2\log x + 3\log(x+1) - \log(x^2 - 1)$.

3. If $\log 2 = 0.3010$, $\log 3 = 0.4771$ and $\log 5 = 0.6990$, then find the values of the following without using table.

(i) $\log 15$ (ii) $\log 64$ (iii) $\log\sqrt{5\times 2}$ (iv) $\log 48$

(v) $\log\sqrt{18}$ (vi) $\log 30$ (vii) $\log\frac{8}{3}$ (viii) $\log\frac{5}{\sqrt{3}}$

4. Prove the following:

(i) $\log_b m \times \log_m a = \log_b a$ (ii) $\log_a b \times \log_c a = \log_c b$

(iii) $\log_b a . \log_c b \frac{1}{\log_c a} = 1$ (iv) $\log_a b = \frac{1}{\log_b a}$

5. Verify the following:

(i) $\log_5 7 \times \log_7 25 = 2$ (ii) $\log_3 2 \times \log_2 81 = 4$

(iii) $\log_5 343 \times \log_7 25 = 6$ (iv) $\log_6 16 \times \log_2 216 = 12$

2.5 Application of Laws of Logarithm

2.5.1 Apply laws of logarithm to convert lengthy processes of multiplication, division and exponential into easier process of addition and subtraction etc.

The following examples will help to understand the application of laws of logarithm.

Example 01 Find the value of (8.573)(28.74) by using logarithm.

Solution:

Let $x = (8.573)(28.74)$

Taking log on both the sides, we have,

$\therefore \quad \log x = \log(8.573 \cdot 28.74)$

$\Rightarrow \quad \log x = \log(8.573) + \log(28.74)$

$\Rightarrow \quad \log x = 0.9332 + 1.4585$

$\Rightarrow \quad \log x = 2.3917$

$\therefore \quad x = \text{antilog}(2.3917)$

Thus, $x = 246.4$

Example 02 Find the value of $\dfrac{213.1}{34.22}$ by using logarithm.

Solution: Let $x = \dfrac{213.1}{34.22}$

Taking log on both the sides, we have,

$$\log x = \log\left(\frac{213.1}{34.22}\right)$$

$$\Rightarrow \quad \log x = \log\left(\frac{213.1}{34.22}\right)$$

$$\Rightarrow \quad \log x = \log 213.1 - \log 34.22, \quad \left(\log\left(\frac{a}{b}\right) = \log a - \log b \right)$$

By referring log table, we have,

$\log x = 2.3286 - 1.5343 = 0.7943$

$\Rightarrow \quad \log x = 0.7943,$

by antilog, we have,

$x = \text{antilog}(0.7943)$, by referring antilog table we have,

$x = 6.227,$ $\quad$ ($\because$ characteristic = 0 and mantissa = 0.7943).

Thus, required value of $\dfrac{213.1}{34.22}$ is found 6.227.

Example 03 Calculate $\sqrt{\frac{3.41\times 37.92}{2.34}}$ by using logarithmic rules.

Solution: Let $x=\sqrt{\frac{3.41\times 37.92}{2.34}}$

Take log on both sides,

$$\log x=\log\left(\frac{3.41\times 37.92}{2.34}\right)^{\frac{1}{2}}$$

$$=\frac{1}{2}\log\left(\frac{3.41\times 37.92}{2.34}\right)$$

$$=\frac{1}{2}(\log 3.41+\log 37.92-\log 2.34)$$

$$=\frac{1}{2}(0.5325+1.5788-0.3692)$$

$$=\frac{1}{2}(1.7424)$$

$$=0.8712$$

$$x=\text{antilog }(0.8712)$$

$$=7433$$

$$=7.433$$

Example 04 Find the number of digits in 4^5

Solution: Let $n=4^5$

Taking log on both the sides, we have,

$\therefore$ $\log n=\log 4^5$, $(\because \log a^n=n\log a)$

$\Rightarrow$ $\log n=5\log 4$,

$\Rightarrow$ $\log n=5\times 0.6021$, (since $\log 4=0.6021$)

$\Rightarrow$ $\log n=3.0105$,

Since number of digits = characteristic +1,

so, number of digits in 4^5= 3+1=4.

Exercise 2.6

1. **Find the values of the following by using logarithms.**

(i) 57.86×4.385

(ii) $25.753 \times 0.5341 \times 490.8$

(iii) $\dfrac{25.753}{0.5341}$

(iv) $\dfrac{(790.6 \times 30.32)}{25.753}$

(v) $\dfrac{99.87}{(8.369) \times (0.785)}$

(vi) $\sqrt[5]{2.709} \times \sqrt[7]{1.239}$

(vii) $\dfrac{(26.62)^{\frac{1}{2}} \times (87.19)^3}{\sqrt{69.53}}$

(viii) $\dfrac{(4308)^3 \times \sqrt{80.06}}{(0.3387)^3}$

2. **Find the number of digits in the following.**

(i) 4^{12} **(ii)** 7^{25} **(iii)** 3^{30} **(iv)** 5^{20} **(v)** 9^{30}

Review Exercise 2

1. Read the following sentences carefully and encircle "T" in case of True and "F" in case of False statement.

(i) 0.025 can be written in scientific notation as 2.5×10^{3} T/F

(ii) Logarithm was invented by Al- Beruni. T/F

(iii) Integral part in the logarithm of a number is called its characteristic. T/F

(iv) Mantissa in the logarithm of a number can be negative. T/F

(v) $\log_a x = y \Leftrightarrow a^y = x$. T/F

2. Fill in the blanks.

(i) Logarithms having base 10 is called __________ .

(ii) $\text{Log}1 =$ __________ .

(iii) Fractional part of logarithm is called __________ .

(iv) $\log_2 512 =$ __________ .

(v) $\log_a m\times\log_m n=$__________ .

(vi) The exponential form of $x = \log_a y$ is __________.

(vii) The logarithmic form of $a^{10}=y$ is__________.

(viii) $\log_b a\times\log_a b=$__________ .

(ix) $\log_a\left(\frac{m}{n}\right)=$ __________.

(x) $\log(10\times10)=$__________ .

(xi) If b>0 then $\log_b 1$ = __________.

(xii) Suppose $\log_b x = \bar{5}.2374$ then its characteristic is __________.

3. Tick (✔) the correct answers.

(i) If $\log_{10} x = 4$, then $x =$ __________ .

(a) 500 (b) 100 (c) 1000 (d) 10000

(ii) The characteristic of log 54.58 is__________ .

(a) 0 (b) 1 (c) 2 (d) 4

(iii) The base of common logarithm is__________ .

(a) 5 (b) 10 (c) e (d) 100

(iv) $\log xyz =$ __________ .

(a) $\log x \log y \log z$ (b) $\log x + \log y + \log z$

(c) $\log (xy)^z$ (d) $\log x - \log y - \log z$

(v) Scientific notation of 0.00789 is __________ .

(a) 7.89×10^{-3} (b) 7.89×10^{3}

(c) 0.789×10^{-2} (d) 78.9×10^{-4}

(vi) If $\log x = 2$ then $x =$__________ .

(a) 200 (b) 1000 (c) 100 (d) $\frac{2}{10}$

(vii) If $\log_2 8 = x$ then $x =$__________ .

(a) 64 (b) 3^2 (c) 3 (d) 2^8

(viii) Base in the Natural logarithm is __________ .

(a) 10 (b) e (c) π (d) 5

(ix) $3^5 = 243$, can be written in logarithmic form as__________ .

(a) $\log_3 5 = 243$ (b) $\log_3 243 = 5$

(c) $\log_5 243 = 12$ (d) $\log_5 3 = 243$

(x) If b>0 and b ≠ 1 then $\log_b \sqrt{b}$ =__________.

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) 2

Summary

- If $a^x = y$, then x is called the logarithm of y to the base 'a' and is written as $\log_a y = x$, where $a > 0, y > 0$ and $a \neq 1$
- Common logarithms have base **10,** it is also named as ***Brigg's*** logarithm and usually written as $\log x$ instead of $\log_{10} x$,natural logarithms have base e, (an irrational number) whose value is 2.7182818…. and written as $\ln x$ instead of $\log_e x$.
- $\log x = y \Leftrightarrow 10^y = x$
- $\ln x = y \Leftrightarrow e^y = x$.
- The integral part of logarithm of any number is called characteristic and fractional part is called mantissa.
- Characteristic of logarithm of a number > 1 is always positive.
- Characteristic of logarithm of a number < 1 is always negative.
- Negative characteristic of logarithm can be written as $\bar{3}, \bar{2}$ *or* $\bar{1}$ instead of –3, –2 or –1.
- The logarithms of a number having the same sequence of digits have same mantissa.
- The number corresponding to a given log is called anti-logarithm.
- Laws of logarithms

(i) $\log_a(mn) = \log_a m + \log_a n$

(ii) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n \log_a m$

(iv) $\log_a n = \dfrac{\log_b n}{\log_b a}$

this law can also be written as $\log_b a . \log_a n = \log_b n$.

Unit 3

• Weightage = 9%

ALGEBRAIC EXPRESSION AND FORMULAS

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Know that a rational expressions behaves like a rational numbers.
- Define a rational expression as a quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$, is not the zero polynomial.
- Examine whether a given algebraic expression is a
  - Polynomial or not,
  - Rational expression or not.
- Define $\frac{p(x)}{q(x)}$ as a rational expression in its lowest form, if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor.
- Examine whether a given rational algebraic expression is in its lowest form or not.
- Reduce a given rational expression to its lowest form.
- Find the sum, difference and the product of rational expressions.
- Divide a rational expression by another rational expression and express the result in its lowest form.
- Find the values of the algebraic expressions at some particular real numbers.
- Know the formulas
  - $(a+b)^2+(a-b)^2=2(a^2+b^2)$ and $(a+b)^2-(a-b)^2=4ab$.
  - Find the values of a^2+b^2 and of ab when the values of $a+b$ and $a-b$ are known.
- Know the formula
  $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$.
  - Find the value of $a^2+b^2+c^2$ when the values of $a+b+c$ and $ab+bc+ca$ are given.
  - Find the value of $a+b+c$ when the values of $a^2+b^2+c^2$ and $ab+bc+ca$ are given.
  - Find the value of $ab+bc+ca$ when the values of $a^2+b^2+c^2$ and $a+b+c$ are given.

- Know the formulas
    - $(a \pm b)^3 = a^3 \pm 3a^2b \pm 3ab^2 \pm b^3$ or
    - $(a \pm b)^3 = a^3 \pm b^3 \pm 3ab(a \pm b)$.
    - Find the values of $a^3 \pm b^3$, when the values of $a \pm b$ and ab are given.
    - Find the values of $x^3 \pm \frac{1}{x^3}$ when the values of $x \pm \frac{1}{x}$ is given.
- Know the formula

  $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.
    - Find the product of $x + \frac{1}{x}$ and $x^2 + \frac{1}{x^2} - 1$.
    - Find the product of $x - \frac{1}{x}$ and $x^2 + \frac{1}{x^2} + 1$.
    - Find the continued product of

      $(x + y)(x - y)(x^2 + xy + y^2))(x^2 - xy + y^2)$.
- Recognize the surds and their applications.
- Explain the surds of the second order.
- Use basic operations on surds of second order to rationalize the denominators and to evaluate them.
- Explain rationalization (with precise meaning) of real numbers of the types $\frac{1}{a + b\sqrt{x}}$, $\frac{1}{\sqrt{x} + \sqrt{y}}$ and their combinations, where x and y are natural numbers and a and b are integers.

3.1 Algebraic Expressions

We have already studied about Algebraic expression in previous classes. Let's discuss its types.

Following are the three types of algebraic expressions.

(*a*) Polynomial Expression or polynomial,

(*b*) Rational Expression,

(*c*) Irrational Expression.

(*a*) Polynomial Expression or polynomial.

A polynomial expression (simply say polynomial) in one variable x can be written as:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} \ldots + a_{n-1}x^1 + a_n$$

Where 'n' is a non–negative integer and the coefficients; $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers. Usually, a polynomial is denoted by $p(x)$, so the above polynomial can be expressed as:

$$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} \ldots + a_{n-1}x^1 + a_n$$

If $a_0 \neq 0$, then the polynomial is said to be a polynomial of degree n, and a_0 is called the **leading coefficient** of the polynomial.

Some examples of polynomials and their degrees are given below.

(i) $8x - 5$, degree 1 (ii) $x^4 - 2x^3 + 5x^2 + 1$, degree 4

(iii) $6x^{31} + 3$, degree 31 (iv) $12x^4 - x^3 + \frac{2}{3}x^2 - 3x + 1$, degree 4

(v) 4, degree zero. (vi) $\sqrt{10}x^{12} + 2x^6 - x^5 - 18x + 1$ degree 12

The algebraic expression $x^3 - x^3y^2 + x^2y^2 - 10$ is a polynomial with two variables x and y and its degree is 5.

Similarly, the algebraic expression $x^3y^5x^2 - x^3y^2z^3 + x^2yz - 34$ is a polynomial with three variables x, y and z and having degree 10(highest sum of powers) and so on.

Remember

- A polynomial consisting of only single term is called monomial. $3x$, $7xy$, $6xy^2z^5$ etc. are some examples of monomials.
- A polynomial consisting of two terms is called binomial e.g. $x+4$, $5x+y$, $7x-3$ etc. are some examples of binomials.
- A polynomial consisting of three terms is called trinomial e.g. x^2-2x+1

$\frac{1}{\sqrt{3}}x^2y^2 - 5xy + 3$, etc. are some examples of trinomials.

- Other polynomial which consisting of four or more terms, called multinomial.
- The highest power (sum of powers in case when more variables are multiplied) on the variable in a polynomial is called the degree of the polynomial.

(b) Rational Expression·

An algebraic expression which can be written in the form $\frac{p(x)}{q(x)}$ where $q(x) \neq 0$, and $p(x)$ and $q(x)$ are polynomials, called a **rational expression** in x.

For example, $\frac{x+1}{x}, \frac{x^2-x+1}{x-5}, \frac{\sqrt{3}x^2-5x+4}{x^2+6x-5}$ etc. are some examples of rational expressions.

Note: Every polynomial is a rational expression but its converse is not true.

(c) Irrational Expression.

An algebraic expression which cannot be written in the form of $\frac{p(x)}{q(x)}$, where $q(x) \neq 0$, when $p(x)$ and $q(x)$ are polynomials is called **irrational expression** in x.

For example, $\frac{1}{\sqrt{x}}, \frac{\sqrt{x}+1}{x}, \frac{\sqrt{x^3}+2x+3}{\sqrt{x}-9}, \sqrt{x}+\frac{5}{\sqrt{x}}$ etc. are some examples of irrational expressions.

3.1.1 Know that a rational expressions behaves like a rational numbers

Let p and q be two integers, then $\frac{p}{q}$ may be an integer or not. Therefore, the number system is extended and $\frac{p}{q}$ is defined as a rational number, where $p, q \in Z$ provided that $q \neq 0$.

Similarly, if $p(x)$ and $q(x)$ are two polynomials, then $\frac{p(x)}{q(x)}$ is not necessarily a polynomial, where $q(x) \neq 0$. Therefore, it is similar to the idea of rational numbers; the concept of rational expressions is developed.

3.1.2 Define a Rational expression as a quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is not the zero polynomial.

As we know that the expression in the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are two polynomials provided $q(x)$ is non–zero polynomial; called a rational expression.

For examples $\frac{x^2-5}{3x^2+4}, 3x^2+4 \neq 0$ are rational expressions.

3.1.3 Examine whether a given algebraic expression is a,

(i) Polynomial or not (ii) Rational expression or not

The following examples will help to identify polynomial and rational expressions.

Example 01 Examine whether the following are the polynomials or not?

(i) $2x^2 - \frac{1}{\sqrt{x}}$ (ii) $6x^3 - 4x^2 - 5x$

Solution(i): $2x^2 - \frac{1}{\sqrt{x}}$

It's not a polynomial because the second term does not have positive integral exponent.

Solution(ii): $6x^3 - 4x^2 - 5x$,

It is a polynomial, because each term has positive integral exponent.

Example 02 Examine whether the following are rational expressions or not?

(i) $\frac{x-2}{3x^2+1}$ (ii) $6x^3 - \frac{1}{\sqrt{x+4}}$

Solution (i): $\dfrac{x-2}{3x^2+1}$

The numerator and denominator both are polynomials, so it is a rational expression.

Solution (ii): $6x^3 - \dfrac{1}{\sqrt{x+4}}$

It is not a rational expression, because the denominator of the second term is not a polynomial.

3.1.4 Define $\dfrac{p(x)}{q(x)}$ as a rational expression in its lowest form, if $p(x)$ and $q(x)$ are polynomials with integral coefficients and having no common factor

The rational expression $\dfrac{p(x)}{q(x)}$ is said to be in its lowest form, if $p(x)$ and $q(x)$ are polynomials with integral coefficients and have no common factor.

For example $\dfrac{x+1}{x-1}$ is the lowest form of $\dfrac{x^2-1}{(x-1)^2}$

3.1.5 Examine whether a given rational algebraic expression is in its lowest form or not

To examine the rational expression $\dfrac{p(x)}{q(x)}$, find common factor(s) of $p(x)$ and $q(x)$. If common factor is 1, then the rational expression is in the lowest form.

For example $\dfrac{x+1}{x-1}$ is in its lowest form because, the common factor of $(x+1)$ and $(x-1)$ is 1.

3.1.6 Reduce a rational expression to its lowest form

Let $\frac{p(x)}{q(x)}$ be the rational expression, where $q(x) \neq 0$.

Step–1: Find the factors of polynomials $p(x)$ and $q(x)$ if possible.

Step–2: Find the common factors of $p(x)$ and $q(x)$.

Step–3: Cancel the common factors of $p(x)$ and $q(x)$.

Example 01 Reduce the following rational expression to their lowest form.

(i) $\frac{(x^2-x)(x^2-5x+6)}{2x(x^2-3x+2)}$ (ii) $\frac{5(x^2-4)}{(3x+6)\,(x-3)}$

Solution (i):

$$\frac{(x^2-x)(x^2-5x+6)}{2x(x^2-3x+2)}$$

$$=\frac{x(x-1)}{2x}\cdot\frac{x^2-3x-2x+6}{x^2-2x-x+2} \qquad \text{(Provided } x\neq 0\text{)}$$

$$=\left(\frac{x-1}{2}\right)\cdot\frac{\{x(x-3)-2(x-3)\}}{\{x(x-2)-1(x-2)\}}$$

$$=\frac{(x-1)(x-3)(x-2)}{2(x-2)(x-1)} \qquad \text{(Provided } x\neq 1 \text{ and } x\neq 2\text{)}$$

$$=\frac{(x-3)}{2}$$

$$=\frac{1}{2}(x-3) \text{ which is the required lowest form}$$

Solution (ii):

$$\frac{5(x^2-4)}{(3x+6)\,(x-3)}$$

$$=\frac{x^2-4}{x-3}\cdot\frac{5}{3x+6}$$

$$=\frac{x^2-2^2}{x-3}\cdot\frac{5}{3(x+2)}$$

$$= \frac{(x+2)(x-2)}{x-3} \cdot \frac{5}{3(x+2)} \qquad \text{(provided } x \neq -2\text{)}$$

$$= \frac{5(x-2)}{3(x-3)} \text{ which is the required lowest form.}$$

3.1.7 Find the sum, difference and product of rational expressions.

The sum, difference and product of rational expression is explained with the help of the following examples.

Example 01 Simplify $\frac{3}{x+1} + \frac{4x}{x^2-1}$

Solution:

$$\frac{3}{x+1} + \frac{4x}{x^2-1}$$

$$= \frac{3}{x+1} + \frac{4x}{(x-1)(x+1)} \quad \text{(Factorization)}$$

$$= \frac{3(x-1)+4x}{(x-1)(x+1)}$$

$$= \frac{3x-3+4x}{(x-1)(x+1)}$$

$$= \frac{7x-3}{(x-1)(x+1)}$$

$$= \frac{7x-3}{x^2-1} \qquad \text{Hence simplified in the lowest form.}$$

Example 02 Simplify $\frac{1}{x^2-1}-\frac{1}{x^3-1}$

Solution:

$$\frac{1}{x^2-1}-\frac{1}{x^3-1}$$

$$=\frac{1}{x^2-1}-\frac{1}{x^3-1}$$

$$=\frac{1}{(x-1)(x+1)}-\frac{1}{(x-1)(x^2+x+1)}$$

$$=\frac{(x^2+x+1)-(x+1)}{(x+1)(x-1)(x^2+x+1)}$$

$$=\frac{x^2+x+1-x-1}{(x+1)(x-1)(x^2+x+1)}$$

$$=\frac{x^2}{(x+1)(x^3-1)}$$

Hence simplified in the lowest form.

Example 03 Simplify $\frac{x^2}{x^2+x-12}\cdot\frac{x^2-9}{2x^2}$

Solution: Simplification

$$\frac{x^2}{x^2+x-12}\cdot\frac{x^2-9}{2x^2}$$

$$=\frac{x^2}{x^2+4x-3x-12}\cdot\frac{(x-3)(x+3)}{2x^2}$$

$$=\frac{1}{x(x+4)-3(x+4)}\cdot\frac{(x-3)(x+3)}{2} \quad \text{(factorization)}$$

$$=\frac{1}{(x+4)(x-3)}\cdot\frac{(x-3)(x+3)}{2} \quad \text{provided } x\neq 3$$

$$=\frac{(x+3)}{2(x+4)}$$

Hence simplified in the lowest form.

3.1.8 Divide a rational expression by another rational expression and express the result in its lowest form.

In order to divide one rational expression by another, we first convert division into multiplication and then simplify the resulting product to lowest form.

Example 01 Simplify $\frac{3x-9y}{2x+10y} \div \frac{x^2-3xy}{4x+20y}$

Solution: Simplification

$$\frac{3x-9y}{2x+10y} \div \frac{x^2-3xy}{4x+20y}$$

$$= \frac{3x-9y}{2x+10y} \times \frac{4x+20y}{x^2-3xy} \quad \text{(Taking Reciprocal)}$$

$$= \frac{3(x-3y)}{2(x+5y)} \times \frac{4(x+5y)}{x(x-3y)}$$

$$= \frac{6}{x}$$

3.1.9 Find the values of algebraic expression at some particular real numbers.

Finding the values of algebraic expressions at some particular real numbers is explained in the following example.

Example 01 Find the value of $\frac{x^2+yz}{x^3+y^2-7yz^4}$ when $x = 3$, $y = 2$ and $z = -1$.

Given $x = 3, y = 2, z = -1$

$$= \frac{x^2+yz}{x^3+y^2-7yz^4}$$

$$= \frac{(3)^2+(2)(-1)}{(3)^3+(2)^2-7(2)(-1)^4}$$

$$= \frac{9-2}{27+4-14}$$

$$= \frac{7}{17}$$

Exercise 3.1

1. Examine whether the following algebraic expressions are polynomials or not.

(i) $2xy^2 - 3x^2 + 5y^3 - 6$ (ii) $3xy^{-2}$

(iii) $6x^2 - 10x + 7 - \sqrt{45}$ (iv) $5\sqrt{x} - x + 5x^2$

(v) $\dfrac{2}{x+2}$ (vi) $\dfrac{2}{x} + x^3 - 2$

2. Examine whether the following algebraic expressions are rational or not

(i) $\dfrac{x^2+2x+3}{x-4}$ (ii) $\dfrac{x^2+5\sqrt{x}-2x}{3x^2+5x+4}$ (iii) $\dfrac{13x^2-9x+4}{x^2+5x+\sqrt{7}}$

(iv) $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ (v) $\dfrac{7}{x+7}$ (vi) $5\sqrt{x} - x + 5x^2$

3. Reduce the following into their lowest form.

(i) $\dfrac{p^2-100}{p+10}$ (ii) $\dfrac{3a^2+3ab}{3a^2+6ab+3b^2}$ (iii) $\dfrac{(a-b)}{(a+b)} \times \dfrac{(a^2+ab)}{(2a^2-2b^2)}$

(iv) $\dfrac{(x+y)^2-z^2}{x+y+z}$ (v) $\dfrac{(m^2-6m)(3m+15)}{2m-12}$ (vi) $\dfrac{x^2-2x-3}{x^2-x-2}$

4. Simplify:

(i) $\dfrac{4x-1}{2x-2} + \dfrac{4x+1}{2x+2}$ (ii) $\dfrac{1}{x+2} + \dfrac{2}{x+3}$ (iii) $\dfrac{xy}{xy+1} + \dfrac{xy+1}{xy-1}$

(iv) $\dfrac{x-2}{x+3} - \dfrac{x+1}{x+6}$ (v) $\dfrac{1}{a+b} - \dfrac{1}{a-b}$ (vi) $\dfrac{4y}{y^2-1} - \dfrac{y+1}{y-1}$

5. Simplify into lowest form.

(i) $\left(\dfrac{x^2}{4y^2-x^2}+1\right) \div \left(1-\dfrac{x}{2y}\right)$ (ii) $\dfrac{x+3}{3y-2x} \cdot \dfrac{4x^2-9y^2}{xy+3y}$

(iii) $\left(\frac{x^2-1}{x^2+2x+1}\times\frac{x+1}{x-1}\right)$ (iv) $\frac{8(y+3)}{9}\times\frac{12(y+1)}{4(y+3)}\div\frac{8(y+1)}{5}$

(v) $\frac{q^2-25}{q^2-3q}\div\frac{q^2+5q}{q^2-9}$ (vi) $\frac{4}{z^2-4z-5}\div\frac{2}{4z^2-4}$

6. Find the value of $t+\frac{1}{t}$, when $t=\frac{x-y}{x+y}$

7. Find the values of

(i) $\frac{5(x+y)}{3x^2\sqrt{y}+6}$, if $x=-4$, $y=9$

(ii) $\frac{42ab^2c^3}{3a^2b+1}$, if $a=3$, $b=2$ and $c=1$

(iii) $\frac{(x+y)^3-z^2}{x^2y^2+z^2}$, if $x=2$, $y=-4$ and $z=3$,

(iv) $\frac{3x^2y}{z}-\frac{bc}{x+1}$, if $x=2$, $y=-1$, $z=3$, b= 4, $c=\frac{1}{3}$

(v) $\frac{(ab^2-c)}{(a+cd^2)}\times\frac{(c+d)}{(a^2b-d)}$, if $a=1$, $b=3$, $c=-3$ and $d=2$.

3.2 Algebraic Formulas

We have already studied and used some algebraic formulas in previous classes. In this section we will learn some more formulas and their applications.

Recall that

- $(a+b)^2=a^2+2ab+b^2$
- $(a-b)^2=a^2-2ab+b^2$
- $(a+b)(a-b)=a^2-b^2$

3.2.1 Know the Formulas

(i) $(a+b)^2+(a-b)^2=2(a^2+b^2)$

Proof:

$$\begin{aligned}\text{L.H.S} &= (a+b)^2+(a-b)^2\\ &= a^2+2ab+b^2+a^2-2ab+b^2 \qquad \because (a+b)^2=a^2+2ab+b^2 \text{ and } (a-b)^2=a^2-2ab+b^2\\ &= 2a^2+2b^2\\ &= 2(a^2+b^2)=\text{R.H.S}\end{aligned}$$

Hence proved

(ii) $(a+b)^2 - (a-b)^2 = 4ab$

Proof:

$$\text{L.H.S} = (a+b)^2 - (a-b)^2$$

$$= a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) \quad \left[\because (a+b)^2 = a^2 + 2ab + b^2\right.$$

$$= a^2 + 2ab + b^2 - a^2 + 2ab - b^2 \quad \left.\text{and } (a-b)^2 = a^2 - 2ab + b^2\right]$$

$$= 4ab = \text{R.H.S}$$

Hence proved

The use of above formulae are explained in the following examples.

Example 01 Find the values of (i) $a^2 + b^2$ (ii) ab (iii) $8ab(a^2 + b^2)$

when $a + b = 6$ and $a - b = 4$.

Solution: Given that,

$a + b = 6,\ a-b = 4.$

(i) $a^2 + b^2 = ?$

We know that, $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

By substituting the values of $a+b = 6$ and $a - b = 4$, we get

$(6)^2 + (4)^2 = 2(a^2 + b^2)$,

$\Rightarrow$ $36 + 16 = 2(a^2 + b^2)$

$\Rightarrow$ $52 = 2(a^2 + b^2)$

$\Rightarrow$ $26 = a^2 + b^2$

$\Rightarrow$ $\boxed{a^2 + b^2 = 26}$

(ii) $ab = ?$

We also know that $(a + b)^2 - (a - b)^2 = 4ab$.

By substituting the values of $a + b = 6$ and $a - b = 4$, we get

$\therefore$ $(6)^2 - (4)^2 = 4ab$

$\Rightarrow$ $36 - 16 = 4ab$

$\Rightarrow$ $20 = 4ab$

$\Rightarrow$ $5 = ab$

or $\boxed{ab = 5}$

(iii) Now $8ab(a^2 + b^2) = 4ab \times 2(a^2 + b^2)$

$= 4(5) \times 2(26)$

$= 20 \times 52$

$\boxed{8ab(a^2 + b^2) = 1040}$

3.3.2 Know the formula

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Proof: L.H.S $= (a+b+c)^2 = (a+b+c)(a+b+c)$

$= a(a+b+c) + b(a+b+c) + c(a+b+c)$

$= a^2 + ab + ac + ab + b^2 + bc + ac + bc + c^2$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca =$ R.H.S Hence proved

The use of this formula is explained in the following examples.

Example 01 Find the value of $a^2+b^2+c^2$, when $a+b+c=7$ and $ab+bc+ca=15$

Solution: Given that,

$a+b+c=7$ and $ab+bc+ca=15$

$a^2+b^2+c^2=?$

We know that $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

Now, substituting the values of $a+b+c=7$ and $ab+bc+ca=15$, in the above formula we get,

$\therefore \quad (7)^2 = a^2+b^2+c^2+2(15)$

$\Rightarrow \quad 49 = a^2+b^2+c^2+30$

$\Rightarrow \quad 49-30 = a^2+b^2+c^2$

$\Rightarrow \quad 19 = a^2+b^2+c^2$

$\Rightarrow \quad \boxed{a^2+b^2+c^2=19}$

Hence, the value of $(a^2+b^2+c^2)$ is 19.

Example 02 Find the value of $(a+b+c)$, when $a^2+b^2+c^2=38$ and $ab+bc+ac=31$

Solution: Given that,

$a^2+b^2+c^2=38$ and $ab+bc+ac=31$,

We know that $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

Now, substituting the values of $a^2+b^2+c^2=38$ and $ab+bc+ac=31$, in the above formula, we get,

$(a+b+c)^2 = 38+2(31)$

$\Rightarrow \quad (a+b+c)^2 = 38+62$

$\Rightarrow \quad (a+b+c)^2 = 100$

$\Rightarrow \quad \sqrt{(a+b+c)^2} = \pm\sqrt{100}$

$\Rightarrow \quad \boxed{(a+b+c)=\pm 10}$

Hence, the value of $(a+b+c)$ is ± 10.

Example 03 Find the value of $(ab+bc+ac)$, when $a+b+c = 8$ and $a^2 + b^2 + c^2 = 20$.

Solution: Given that,

$a+b+c = 8$ and $a^2 + b^2 + c^2 = 20$,

We know that $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$

By substituting the values of $a + b + c = 8$ and $a^2 + b^2 + c^2 = 20$, in the above formula we get,

$(8)^2 = 20 + 2(ab + bc + ac)$

$\Rightarrow \quad 64 = 20 + 2(ab + bc + ac)$

$\Rightarrow \quad 64 - 20 = 2(ab + bc + ac)$

$\Rightarrow \quad 44 = 2(ab + bc + ac)$

$\Rightarrow \quad 22 = ab + bc + ac$

$\Rightarrow \quad ab + bc + ac = 22$

Hence, the value of $(ab + bc + ac)$ is 22.

3.2.3 Know the cubic formulas

(i) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

or $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

Proof: L.H.S $= (a+b)^3 = (a+b)(a+b)^2$

$= (a+b)(a^2+2ab+b^2)$

$= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3$

$= a^3+3a^2b+3ab^2+b^3$

$= a^3+b^3+3ab(a+b)$ = R.H.S

Hence proved

(ii) $(a - b)^3 = a^3 - b^3 + 3ab(a - b)$

or $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

Proof: L.H.S $= (a-b)^3 = (a-b)(a-b)^2$

$= (a-b)(a^2-2ab+b^2)$

$= a^3-2a^2b+ab^2-a^2b+2ab^2-b^3$

$= a^3-3a^2b+3ab^2-b^3$

$= a^3-b^3-3ab(a-b)$ = R.H.S

Hence proved

The following examples are helpful for understanding the application for Cubic formulas.

Example 01 Find the value of $a^3 + b^3$, when $a + b = 4$ and $ab = 5$.

Solution: Given that,

$a + b = 4$ and $ab = 5$

We have to find

$a^3 + b^3$

Since, $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$.

By substituting the values of $a + b = 4$ and $ab = 5$, in the above formula we get,

$(4)^3 = a^3 + b^3 + 3(5)(4)$

$\Rightarrow \quad 64 = a^3 + b^3 + 60$

$\Rightarrow \quad 64 - 60 = a^3 + b^3$

$\Rightarrow \quad 4 = a^3 + b^3$

$\Rightarrow \quad \boxed{a^3 + b^3 = 4}$

Hence the value of $(a^3 + b^3)$ is 4.

Example 02 Find the value of ab, when $a^3 - b^3 = 5$ and $a - b = 5$.

Solution: Given that,

$a^3 - b^3 = 5$ and $a - b = 5$

We have to find ab

Since, $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$.

Now, substituting the values of $a^3 - b^3 = 5$ and $a - b = 5$, in the above formula, we get,

$(5)^3 = 5 - 3ab(5)$

$\Rightarrow \quad 125 = 5 - 15ab$

$\Rightarrow \quad 125 - 5 = -15ab$

$\Rightarrow \quad 120 = -15ab$

$\Rightarrow \quad -8 = ab$

$\Rightarrow \quad \boxed{ab = -8}$

Example 03 Find the value of $x^3+\frac{1}{x^3}$ when $x+\frac{1}{x}=3$

Solution: Given that

$$x+\frac{1}{x}=3$$

Taking cube on both sides, we have

$$\left(x+\frac{1}{x}\right)^3=3^3$$

$$\Rightarrow \quad x^3+\frac{1}{x^3}+3x\cdot\frac{1}{x}\left(x+\frac{1}{x}\right)=27 \qquad [\because (a+b)^3=a^3+b^3+3ab(a+b)]$$

$$\Rightarrow \quad x^3+\frac{1}{x^3}+3(3)=27$$

$$\Rightarrow \quad x^3+\frac{1}{x^3}+9=27$$

$$\Rightarrow \quad x^3+\frac{1}{x^3}=27-9$$

$$\Rightarrow \quad \boxed{x^3+\frac{1}{x^3}=18}$$

Example 04 Find $8x^3-\frac{1}{x^3}$, when $2x-\frac{1}{x}=4$

Solution: Given that

As $2x-\frac{1}{x}=4$

Cubing on both sides, we set

$$(2x-\frac{1}{x})^3=(4)^3$$

$$(2x)^3-(\frac{1}{x})^3-3(2x)(\frac{1}{x})(2x-\frac{1}{x})=64$$

$$8x^3-\frac{1}{x^3}-6(4)=64$$

$$8x^3-\frac{1}{x^3}-24=64 \qquad \Rightarrow 8x^3-\frac{1}{x^3}=88$$

3.2.4 Know the formula $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

(i) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Proof: R.H.S $= (a+b)(a^2 - ab + b^2)$

$= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$

$= a^3 + b^3 =$ L.H.S

Hence proved

(ii) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Proof: R.H.S $= (a-b)(a^2 + ab + b^2)$

$= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$

$= a^3 - b^3 =$ L.H.S

Hence proved

Example 01 Find the product of $\left(x+\frac{1}{x}\right)$ and $\left(x^2+\frac{1}{x^2}-1\right)$

Solution: $\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$

$\left(x+\frac{1}{x}\right)\left((x)^2-(x)\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2\right)$

$\because (a+b)(a^2-ab+b^2) = a^3+b^3$

Thus, $\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right) = x^3+\frac{1}{x^3}$

Example 02 Find the product of $\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right) = x^3-\frac{1}{x^3}$

Solution: $\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)$

$\left(x-\frac{1}{x}\right)\left((x)^2+(x)\left(\frac{1}{x}\right)+\left(\frac{1}{x}\right)^2\right)$

$\because (a-b)(a^2+ab+b^2) = a^3-b^3$

Thus, $\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right) = x^3-\frac{1}{x^3}$

Example 03 Find the continued product of:

$(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

Solution: $(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

$= (x^3+y^3)(x^3-y^3)$ $\quad [\because (a \pm b)(a^2 \mp ab + b^2) = a^3 \pm b^3]$

$= (x^3)^2-(y^3)^2$

$= x^6 - y^6$

Exercise 3.2

Find the value of

1. $a^2 + b^2$ and ab, when $a + b = 8$ and $a - b = 6$.
2. $a^2 + b^2$ and ab, when $a + b = 5$ and $a - b = 3$.
3. $a^2 + b^2 + c^2$, when $a + b + c = 9$ and $ab + bc + ac = 13$.
4. $a^2 + b^2 + c^2$, when $a + b + c = \dfrac{1}{3}$ and $ab + bc + ac = \dfrac{-2}{9}$.
5. $a + b + c$, when $a^2 + b^2 + c^2 = 29$ and $ab + bc + ac = 10$.
6. $a + b + c$, when $a^2 + b^2 + c^2 = 0.9$ and $ab + bc + ac = 0.8$.
7. $ab + bc + ac$, when $a + b + c = 10$ and $a^2 + b^2 + c^2 = 20$.
8. $a^3 + b^3$, when $a + b = 4$ and $ab = 3$.
9. ab, when $a^3 - b^3 = 5$ and $a - b = 5$.
10. ab, when $a^3 - b^3 = 16$ and $a - b = 4$.
11. $a^3 - b^3$, when $a - b = 5$ and $ab = 7$.
12. $125x^3 + y^3$ when $5x + y = 13$ and $xy = 10$.
13. $216a^3 - 343b^3$, when $6a - 7b = 11$ and $ab = 8$.
14. $x^3 + \dfrac{1}{x^3}$, when $x + \dfrac{1}{x} = 7$.
15. $x^3 - \dfrac{1}{x^3}$, when $x - \dfrac{1}{x} = 11$
16. Find product of

    (i) $\left(\dfrac{3}{2}b + \dfrac{2}{3b}\right)\left(\dfrac{9b^2}{4} + \dfrac{4}{9b^2} - 1\right)$ (ii) $\left(\dfrac{7y^2}{9} + \dfrac{9}{7y^2}\right)\left(\dfrac{49y^4}{81} + \dfrac{81}{49y^4} - 1\right)$

    (iii) $\left(\dfrac{x^4}{12} + \dfrac{12}{x^4}\right)\left(\dfrac{x^8}{144} + \dfrac{144}{x^8} + 1\right)$ (iv) $\left(c^2 - \dfrac{1}{c^2}\right)\left(c^4 + \dfrac{1}{c^4} + 1\right)$

17. Find the continued product by using the relevant formulas.

    (i) $(2x^2 + 3y^2)(4x^4 - 6x^2y^2 + 9y^4)$

    (ii) $(2x^2 - 3y^2)(4x^4 + 6x^2y^2 + 9y^4)$

    (iii) $(x - y)(x + y)(x^2 + y^2)(x^2 + xy + y^2)(x^2 - xy + y^2)(x^4 - x^2y^2 + y^4)$.

    (iv) $(2x + 3y)(2x - 3y)(4x^2 + 9y^2)(16x^4 + 81y^4)$

3.3 Surds and their Applications

3.3.1 Recognize the surds and their applications.

Surd: An expression is called a surd which has at least one term contain radical term in its simplified form.

For examples, $\sqrt{2}, \sqrt{a-4}, \sqrt[3]{\frac{5}{10}}, \left(\frac{1}{3}+\sqrt{3}\right), \left(\sqrt[5]{2}-\frac{1}{2}\right)$ are surds.

All surds are irrational numbers.

If $\sqrt[n]{a}$ is an irrational number and 'a' is not a perfect n^{th} power then it is called a surd of n^{th} order. The result of $\sqrt[n]{a}$ is an irrational number. It is also called an irrational radical with rational radicand.

For examples: $\sqrt{\frac{5}{7}}, \sqrt[3]{5}, \sqrt[4]{6}, \sqrt[5]{2}, \sqrt[7]{10}$ are surds of order 2nd, 3rd, 4th, 5th and 7th respectively. But $\sqrt[3]{27}$ and $\sqrt{\frac{1}{4}}$ are not surds because they represent the number 3 and $\frac{1}{2}$ respectively.

3.3.2 Explain the surds of the second order use basic operations on surds of second order to rationalize the denominators and to evaluate them.

(a) Surds of the second order:

(i) A surd which contains a single term is called a monomial surds.

For examples, $\sqrt{53}, \sqrt{a-9}, \sqrt{\frac{4}{5}}$ etc. are monomials and of 2nd orders.

(ii) A surd which contains sum or difference of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd.

For examples, $\sqrt{17}+\sqrt{11}, \sqrt{2}-13, \sqrt{3}-35$ etc. are binomial surds and of 2nd order.

(iii) Conjugate of Binomial Surds

Expressions of the type

(a) $\left(\sqrt{a}+c\sqrt{b}\right)$ and $\left(\sqrt{a}-c\sqrt{b}\right)$ are conjugate surds of each other.

(b) $a+\sqrt{b}$ and $a-\sqrt{b}$ are conjugate surds of each other.

(b) Basic operations on surds of second order to rationalize the denominators and to evaluate them.

(i) Addition and subtraction of Surds.

The addition and subtraction of surds can be done by using following law.

For example, $a\sqrt{c}+b\sqrt{c}=(a+b)\sqrt{c}$ and $a\sqrt{c}-b\sqrt{c}=(a-b)\sqrt{c}$

Example 01 Simplify: $\sqrt{343}-3\sqrt{7}-2\sqrt{7}$

Solution:

$$\sqrt{343}-3\sqrt{7}-2\sqrt{7}$$
$$=\sqrt{7\times7\times7}-3\sqrt{7}-2\sqrt{7}$$
$$=7\sqrt{7}-3\sqrt{7}-2\sqrt{7}$$
$$=(7-3-2)\sqrt{7}$$
$$=(7-5)\sqrt{7}$$
$$=2\sqrt{7}$$

Example 02 Simplify: $\sqrt{32}+5\sqrt{2}+\sqrt{128}+7\sqrt{2}$

Solution:

$$\sqrt{32}+5\sqrt{2}+\sqrt{128}+7\sqrt{2}$$
$$=\sqrt{16\times2}+5\sqrt{2}+\sqrt{64\times2}+7\sqrt{2}$$
$$=\sqrt{(4)^2\times2}+5\sqrt{2}+\sqrt{(8)^2\times2}+7\sqrt{2}$$
$$=4\sqrt{2}+5\sqrt{2}+8\sqrt{2}+7\sqrt{2}$$
$$=(4+5+8+7)\sqrt{2}$$
$$=24\sqrt{2}$$

(ii) Multiplication and Division of Surds.

The Multiplication and division of the surds can be simplified by using the following laws:

(a) $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ (b) $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$, provided $a>0$ and $b>0$.

Example 01 Simplify: $\sqrt{125}\times\sqrt{48}$

Solution: Simplification

$$\sqrt{125}\times\sqrt{48}$$
$$=\sqrt{(5)^2\times5}\times\sqrt{(4)^2\times3}$$
$$=5\sqrt{5}\times4\times\sqrt{3}$$
$$=(5\times4)\left(\sqrt{5}\times\sqrt{3}\right)$$
$$=20\sqrt{15}$$

Example 02 Simplify: $\dfrac{\sqrt{162}}{\sqrt{144}}$

Solution:

$$\frac{\sqrt{162}}{\sqrt{144}}$$
$$=\frac{\sqrt{2\times81}}{\sqrt{12\times12}}$$
$$=\frac{\sqrt{2\times(9)^2}}{\sqrt{(12)^2}}=\frac{9\sqrt{2}}{12}=\frac{3\sqrt{2}}{4}$$

Exercise 3.3

1. Simplify

(i) $\sqrt[4]{81x^{-8}z^{4}}$ (ii) $\sqrt[3]{256a^{6}b^{12}c^{9}}$ (iii) $\sqrt[7]{128}$ (iv) $\sqrt{7776}$

(v) $\dfrac{\sqrt[3]{(125)^{2}\times 8}}{\sqrt{(2\times 32)^{2}}}$ (vi) $\dfrac{\sqrt{21}\times\sqrt{28}}{\sqrt{121}}$ (vii) $\sqrt{\dfrac{(216)^{\frac{2}{3}}\times(125)^{2}}{(0.04)^{-3}}}$

(viii) $\dfrac{\sqrt[6]{4}\times\sqrt[3]{27}\times\sqrt{60}}{\sqrt{180}\times\sqrt[3]{0.25}\times\sqrt[4]{9}}$

2. Find the conjugate of

(i) $\left(8-4\sqrt{3}\right)$ (ii) $\left(6\sqrt{6}+2\sqrt{3}\right)$ (iii) $\left(8\sqrt{12}+\sqrt{8}\right)$

(iv) $\left(2-\sqrt{3}\right)$

3. Simplify

(i) $\left(6\sqrt{2}+4\sqrt{2}+7\sqrt{128}\right)$ (ii) $\sqrt{5}+\sqrt{125}+7\sqrt{5}$

(iii) $\left(13+15\sqrt{3}\right)+\left(7-6\sqrt{3}\right)$ (iv) $\sqrt{250}+\sqrt{490}+3\sqrt{10}$

(v) $\sqrt{245}+\sqrt{625}-\sqrt{45}$ (vi) $10\sqrt{11}-\sqrt{396}-3\sqrt{11}$

(vii) $\sqrt{17}\left(10\sqrt{17}-2\sqrt{17}\right)$ (viii) $\dfrac{3}{2}\left(\sqrt{18}+\sqrt{32}-\sqrt{50}\right)$

(ix) $\left(\dfrac{\sqrt{2}}{\sqrt{3}}+\dfrac{1}{\sqrt{3}}\right)\left(\dfrac{\sqrt{2}}{\sqrt{3}}-\dfrac{1}{\sqrt{3}}\right)$ (x) $\left(\sqrt{13}+\sqrt{11}\right)\left(\sqrt{13}-\sqrt{11}\right)$

(xi) $\left(3\sqrt{6}-4\sqrt{5}\right)^{2}$ (xii) $\left(2\sqrt{3}+3\sqrt{2}\right)^{2}$

3.4 Rationalization

3.4.1 Explain rationalization (with precise meaning) of real numbers on surds of the types $\frac{1}{a+b\sqrt{x}}$, $\frac{1}{\sqrt{x}+\sqrt{y}}$ and their combinations, where x,y are natutal number and a and b are integer.

- If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.
  For example, $\left(35+\sqrt{31}\right)$ and $\left(35-\sqrt{31}\right)$ are rationalizing factor of each other.
- The process of multiplying a given surd by its rationalizing factor to get a rational number as product is called rationalization of the given surd. The product of the conjugate surds is a rational number.

Example 01 Find the product of $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

Solution: $\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

$=\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2$

$=3-2=1$ which is a rational number.

Rationalization of denominator

Keeping the above discussion in mind, we observe that, in order to rationalize a denominator of the form $(a+b\sqrt{x})$ or $(a-b\sqrt{x})$, we multiply both numerator and denominator by the conjugate factor $(a-b\sqrt{x})$ or $(a+b\sqrt{x})$, by doing this we eliminate the radical and thus obtain a denominator free of the surd.

Rationalization of real numbers of the Types.

(i) $\frac{1}{a+b\sqrt{x}}$ (ii) $\frac{1}{\sqrt{x}+\sqrt{y}}$

For the expressions $\frac{1}{a+b\sqrt{x}}$ and $\frac{1}{\sqrt{x}+\sqrt{y}}$ also their rationalization, where $x,y\in\mathbf{N}$ and $a,b\in\mathbf{Z}$. The following examples will help to understand the concept of rationalization.

Example 01 Rationalize: $\dfrac{1}{5+2\sqrt{3}}$

Solution: $\dfrac{1}{5+2\sqrt{3}}$

Multiplying and dividing by conjugate of denominator, we have

$$= \frac{1}{5+2\sqrt{3}} \times \frac{5-2\sqrt{3}}{5-2\sqrt{3}}$$

$$= \frac{5-2\sqrt{3}}{(5)^2-(2\sqrt{3})^2}$$

$$= \frac{5-2\sqrt{3}}{25-12}$$

$$= \frac{5-2\sqrt{3}}{13}$$

Observe that the denominator has been obtained free from radical sign due to rationalization. Hence, we obtain a rational number in the denominator. This process is said to be rationalization

Example 02 Rationalize: $\dfrac{5}{\sqrt{3}+\sqrt{2}}$

Solution:

$$\frac{5}{\sqrt{3}+\sqrt{2}} = \frac{5}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

$$= \frac{5\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{5\left(\sqrt{3}-\sqrt{2}\right)}{3-2}$$

$$= \frac{5\left(\sqrt{3}-\sqrt{2}\right)}{1}$$

$$= 5\left(\sqrt{3}-\sqrt{2}\right)$$

Example 03 If $x = 2-\sqrt{3}$ then find the value of $x^2-\dfrac{1}{x^2}$ and $x^2+\dfrac{1}{x^2}$

Solution: As $x = 2-\sqrt{3}$

$$\therefore \quad \frac{1}{x} = \frac{1}{2-\sqrt{3}}$$

$$\frac{1}{x} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$\frac{1}{x} = \frac{2+\sqrt{3}}{(2-\sqrt{3})(2+\sqrt{3})}$$

$$\frac{1}{x} = \frac{2+\sqrt{3}}{4-3} = 2+\sqrt{3}$$

$$\therefore \quad x+\frac{1}{x} = (2-\sqrt{3})+(2+\sqrt{3})$$

$$x+\frac{1}{x} = 4$$

$$\therefore \quad x-\frac{1}{x} = (2-\sqrt{3})-(2+\sqrt{3})$$

$$x-\frac{1}{x} = \cancel{2}-\sqrt{3} - \cancel{2}-\sqrt{3}$$

$$x-\frac{1}{x} = -2\sqrt{3}$$

Now,

$$x^2-\frac{1}{x^2} = \left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$$

$$x^2-\frac{1}{x^2} = 4\left(-2\sqrt{3}\right)$$

$$x^2-\frac{1}{x^2} = -8\sqrt{3}$$

Also, $\left(x+\dfrac{1}{x}\right)^2 = (4)^2$

$$\Rightarrow \quad x^2+2+\frac{1}{x^2} = 16$$

$$\Rightarrow \quad x^2+\frac{1}{x^2} = 16-2 = 14$$

Exercise 3.4

1. **Rationalize the denominator of the following.**

(i) $\frac{1}{2+\sqrt{3}}$ (ii) $\frac{1}{3+2\sqrt{2}}$ (iii) $\frac{1}{4\sqrt{3}-5\sqrt{2}}$

(iv) $\frac{16}{2\sqrt{3}+\sqrt{11}}$ (v) $\frac{9-\sqrt{2}}{9+\sqrt{2}}$ (vi) $\frac{\sqrt{13}+3}{\sqrt{13}-3}$

2. (i) If $x=8-3\sqrt{7}$, find the value of $\left(x+\frac{1}{x}\right)^2$

(ii) If $\frac{1}{x}=2\sqrt{28}-11$, find the value of x.

(iii) If $x=3-2\sqrt{2}$

find the value of: $x+\frac{1}{x}, x-\frac{1}{x}, x^2+\frac{1}{x^2}, x^2-\frac{1}{x^2}$ and $x^4+\frac{1}{x^4}$

3. If $x=\sqrt{5}+2$, find the value of $x^4+\frac{1}{x^4}$.

4. If $\frac{1}{y}=2+\sqrt{3}$, find the value of $y^4+\frac{1}{y^4}$.

5. If $\frac{1}{z}=7-4\sqrt{3}$, find the value of z^2-z^{-2}.

Review Exercise 3

1. Encircle the correct answer.

(i) Every polynomial is:
(a) an irrational expression (b) a rational expression
(c) a sentence (d) none of these

(ii) A surd which contains sum of two monomial surds is called
(a) Trinomial surd (b) Binomial surd
(c) Conjugate surd (d) Monomial surd

(iii) $3x + 2y - 3$ is an algebraic
(a) Expression (b) Equation
(c) Sentence (d) In−equation

(iv) The degree of the $3x^2y + 5y^4 - 10$ is
(a) 4 (b) 5 (c) 6 (d) 3

(v) $\sqrt{7}$ is an example of
(a) Monomial surd (b) Trinomial surd
(c) Binomial surd (d) Conjugate surd

(vi) Quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x) \neq 0$ is called
(a) Rational expression (b) Irrational expression
(c) Polynomial (d) Conjugate

(vii) $\frac{1}{x-y} - \frac{1}{x+y}$ is equal to
(a) $\frac{2x}{x^2-y^2}$ (b) $\frac{2y}{x^2-y^2}$ (c) $\frac{-2x}{x^2-y^2}$ (d) $\frac{-2y}{x^2-y^2}$

(viii) Conjugate of $2 - \sqrt{3}$ is
(a) $2+\sqrt{3}$ (b) $-2-\sqrt{3}$ (c) $\sqrt{2}+3$ (d) $\sqrt{3}-4$

(ix) $a^3 - 3ab(a-b) - b^3$ is equal to
(a) $(a-b)^3$ (b) $(a+b)^3$ (c) a^3+b^3 (d) a^3-b^3

(x) If $a+b=5$ and $a-b=3$, then the value of ab is
(a) 4 (b) 5 (c) 3 (d) 6

(xi) $\left(5+\sqrt{15}\right)\left(5-\sqrt{15}\right)$ is equal to

(a) 10 (b) 15 (c) 25 (d) 30

(xii) $a^2+b^2+c^2+2ab+2bc+2ca$ is equal to

(a) $(a+b-c)^2$ (b) $(a+b+c)^2$

(c) $(a-b+c)^2$ (d) $(a+b+c)^3$

2. Fill in the blanks.

(i) Degree of any polynomial is __________ .

(ii) Conjugate of surd $2-\sqrt{3}$ is __________ .

(iii) Degree of polynomial $2x^3+x^2-4x^4+7x-9$ is __________ .

(iv) $\dfrac{\sqrt{x}}{3x+5}$ is a/an __________ expression.

(v) $(x-y)(x+y)(x^2+y^2)=$__________ .

Summary

- A polynomial expression (simply say polynomial) in one variable x can be written as: $a_0x^n+a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+\cdots+a_{n-1}x^1+a_n$
  A polynomial is usually denoted by $p(x)$.
- An algebraic expression which can be written in the form $\dfrac{p(x)}{q(x)}$, where $q(x)\neq 0$, and $p(x)$, $q(x)$ are both polynomials, called **rational expression** in x.
- An algebraic expression which cannot be written in form of $\dfrac{p(x)}{q(x)}$, where $q(x)\neq 0$, and $p(x)$, $q(x)$ are both polynomials, called irrational expression
- A polynomial expression consisting of only single term is called monomial.
- A polynomial expression consisting of two terms is called binomial.
- A polynomial expression consisting of three terms is called trinomial.
- Polynomial expression consisting two or more than two terms is called multinomial.

- The rational expression $\frac{p(x)}{q(x)}$, is said to be in its lowest form, if $p(x)$ and $q(x)$ are polynomials with integral coefficients and have no common factor.
- $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$ and $(a+b)^2 - (a-b)^2 = 4ab$.
- $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac$.
- $(a+b)^3 = a^3+b^3+3ab(a+b)$ and $(a-b)^3 = a^3-b^3-3ab(a-b)$
- $a^3+b^3 = (a+b)(a^2-ab+b^2)$.
- $a^3-b^3 = (a-b)(a^2+ab+b^2)$.
- An expression is called a surd which has at least one term involving a radical sign. For example, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{\frac{3}{10}}$ are surds.
- If $\sqrt[n]{a}$ is an irrational number and **'*a*'** is not a perfect n^{th} power then it is called a surd of n^{th} order.
- A surd which contains a single term is called a monomial.
- A surd which contains sum or difference of two surds or sum of monomial surd and a rational number is called binomial surd.
- Expressions of the type $\left(\sqrt{a}+c\sqrt{b}\right)$ and $\left(\sqrt{a}-c\sqrt{b}\right)$ are conjugate surds of each other.
- $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$ and $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$, provided $a>0$ and $b>0$.
- If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.
- The process of multiplying a given surd by its rationalizing factor to get a rational number as product is called rationalization of the given surd.

Unit 4

• Weightage = 9%

FACTORIZATION

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Recall factorization of expressions of the following types.
  - $ka + kb + kc$ (Common factors in all the terms)
  - $ac + ad + bc + bd$ (Common factors in pairs of terms)
  - $a^2 \pm 2ab + b^2$ (Perfect squares)
  - $a^2 - b^2$ (Difference of two squares)
  - $(a^2 \pm 2ab + b^2) - c^2$
  - $(\sqrt{a})^2 - (\sqrt{b})^2$
- Factorize the expressions of the following types.

  Type I: $a^4 + a^2 b^2 + b^4$ and $a^4 + b^4$

  Type II: $x^2 + px + q$

  Type III: $ax^2 + bx + c$

  Type IV: $(ax^2 + bx + c)(ax^2 + bx + d) + k$

  $(x + a)(x + b)(x + c)(x + d) + k$

  $(x + a)(x + b)(x + c)(x + d) + kx^2$

  Type V: $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 - 3a^2b + 3ab^2 - b^3$

  Type VI: $a^3 \pm b^3$
- State and prove remainder theorem and explain through examples.
- Find remainder (without dividing) when a polynomial is divided by a linear polynomial.
- Define zero of a polynomial.
- State and prove factor theorem.
- Describe the method of synthetic division.
- Use synthetic division to:
  - Find quotient and remainder when a given polynomial is divided by a linear polynomial.
  - Find the value(s) of unknown(s) if the zeros of the polynomial are given.
  - Find the value(s) of unknown(s) if the factors of the polynomial are given.
- Use factor theorem to factorize a cubic polynomial.

Introduction

we will study about factorization which has an important role in mathematics. It helps us to reduce the complicated expression into simple expressions.

4.1 Factorization

Let $p(x)$,$q(x)$ and $r(x)$ are three polynomials such that, $p(x)\times q(x) = r(x)$. Here, the resulting polynomial $r(x)$ is the product of $p(x)$ and $q(x)$,and the polynomials $p(x)$ and $q(x)$ are called the **factors of $r(x)$**.

There are some examples of factors of the polynomials are given below.

(i) $6x^2y^3=(2\times 3)(x\times x)(y\times y\times y)$

(ii) $ax + aby + abcz = a(x + by + bcz)$

(iii) $5x + 15xy= 5x(1+3y)$

(iv) $x - y=\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2=\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)$

4.1.1 Recall Factorization of Expressions of the Following Types

(i) $ka + kb + kc$ (Common factors in all the terms)

(ii) $ac + ad + bc + bd$ (Common factors in pairs of terms)

(iii) $a^2\pm 2ab + b^2$ (Perfect squares)

(iv) $a^2 - b^2$ (Difference of two squares)

(v) $a^2\pm 2ab + b^2 - c^2$

(vi) $\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2$

(i) Factors of the type: $ka + kb + kc$

Let us see the following examples

Example 01 Factorize: $10a + 15b - 20c$

Solution: $10a + 15b - 20c$

$= 5(2a + 3b - 4c)$ (5 as a common from the expression)

Example 02 Find the factors of $\frac{4}{9}-\frac{8}{12}x-\frac{16}{15}xy$

Solution: $\frac{4}{9}-\frac{8}{12}x-\frac{16}{15}xy$

$=\frac{4}{3\cdot 3}-\frac{4\cdot 2}{3\cdot 4}x-\frac{4\cdot 4}{3\cdot 5}xy$

$=\frac{4}{3}\left(\frac{1}{3}-\frac{2}{4}x-\frac{4}{5}xy\right)$ (Taking $\frac{4}{3}$ as a common from the expression)

(ii) Factors of the type: $ac + ad + bc + bd$

See the following examples

Example 01 Find the factors of $3a - ac - 3c + c^2$

Solution: $3a - ac - 3c + c^2$

$= a(3 - c) - c(3 - c)$

$= (3 - c)(a - c)$

Example 02 $9y^2z + 3xyz - 9xy^2 - 3x^2y$

Solution: $9y^2z + 3xyz - 9xy^2 - 3x^2y$

$= 3yz(3y + x) - 3xy(3y + x)$

$= (3y + x)(3yz - 3xy)$

$= (3y + x) \times 3y(z - x)$

$= 3y(x + 3y)(z - x)$ are the required factors.

(iii) Factors of the type: $a^2 \pm 2ab + b^2$

As we know that,

$$a^2 + 2ab + b^2 = (a)^2 + 2(a)(b) + (b)^2 = (a + b)^2$$

and $$a^2 - 2ab + b^2 = (a)^2 - 2(a)(b) + (b)^2 = (a - b)^2$$

Let us see the following examples

Example 01 Factorize $16a^2 + 40ab + 25b^2$

Solution: $16a^2 + 40ab + 25b^2$

$= (4a)^2 + 2(4a)(5b) + (5b)^2$

$= (4a + 5b)^2$ $[\because a^2+2ab+b^2=(a+b)^2]$

Example 02 Factorize $4p^2 - 28pq + 49q^2$

Solution: $4p^2 - 28pq + 49q^2$

$= (2p)^2 - 2(2p)(7q) + (7q)^2$

$= (2p - 7q)^2$ $[\because a^2 - 2ab + b^2 = (a - b)^2]$

(iv) Factors of the type: $a^2 - b^2$

Let us see following examples

Example 01 Factorize $4x^2 - 1$

Solution: $4x^2 - 1$

$= (2x)^2 - (1)^2$

$= (2x - 1)(2x + 1)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

Example 02 Factorize $96y^2 - 6z^2$

Solution: $96y^2 - 6z^2$

$= 6(16y^2 - z^2)$

$= 6[(4y)^2 - z^2]$

$= 6(4y - z)(4y + z)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

Example 03 Factorize $9r^4 - (6s - t^2)^2$

Solution: $9r^4 - (6s - t^2)^2$

$= (3r^2)^2 - (6s - t^2)^2$

$= [3r^2 - (6s - t^2)][3r^2 + (6s - t^2)],$ $[\because a^2 - b^2 = (a - b)(a + b)]$

$= (3r^2 - 6s + t^2)(3r^2 + 6s - t^2)$

(v) Factors of the type: $(a^2 \pm 2ab + b^2) - c^2$

Let us see following examples.

Example 01 Factorize $x^2 + 4xy^2 + 4y^4 - 4z^2$

Solution: $x^2 + 4xy^2 + 4y^4 - 4z^2$

$= \{(x)^2 + 2(x)(2y^2) + (2y^2)^2\} - (2z)^2$

$= (x + 2y^2)^2 - (2z)^2$

$= \{(x+2y^2) + 2z\}\{(x + 2y^2) - 2z\}$ $[\because a^2 - b^2 = (a - b)(a + b)]$

$= (x + 2y^2 + 2z)(x + 2y^2 - 2z)$

Example 02 $9p^2 - 6pq + q^2 - 9r^2$

Solution: $9p^2 - 6pq + q^2 - 9r^2$

$= (3p)^2 - 2(3p)(q) + q^2 - (3r)^2$

$= (3p - q)^2 - (3r)^2$

$= (3p - q + 3r)(3p - q - 3r)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

(vi) Factors of the type: $\left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2$

Example 01 Factorize $\left(\sqrt{xy}\right)^2 - \left(\sqrt{z}\right)^2$

Solution: $\left(\sqrt{xy}\right)^2 - \left(\sqrt{z}\right)^2$

$= \left(\sqrt{xy} - \sqrt{z}\right)\left(\sqrt{xy} + \sqrt{z}\right)$ $[\because a^2 - b^2 = (a - b)(a + b)]$

Exercise 4.1

1. Factorize the following:

(i) $4x + 16y + 24z$ (ii) $x^2 + 3x^2y + 4x^2y^2z$

(iii) $3pqr + 6pqt + 3pqs$ (iv) $9qr(s^2 + t^2) + 18q^2r^2(s^2 + t^2)$

(v) $\frac{z^2x}{16} - \frac{x^2z^2}{8} + \frac{x^2z^3}{12}$ (vi) $a(x-y) - a^2b(x-y) + a^2b^2(x-y)$

2. Factorize the following:

(i) $7x + xz + 7z + z^2$ (ii) $9a^2b + 18ab^2 - 6ac - 12bc$

(iii) $6t - 12p + 4tq - 8pq$ (iv) $r^2 + 9rs - 7rs - 63s^2$

(v) $\frac{y^2}{4} - \frac{y^2z}{4} - \frac{z^2t}{9} + \frac{z^3t}{9}$ (vi) $\frac{10xy}{11} + \frac{5xz}{11} - \frac{14y^2}{11} - \frac{7yz}{11}$

3. Factorize:

(i) $4a^2 + 12ab + 9b^2$ (ii) $36x^4 + 12x^2 + 1$

(iii) $x^2 + 1 + \frac{1}{4x^2}$ (iv) $81y^2 + 144yz + 64z^2$

(v) $625 + 50\,a^2b + a^4b^2$ (vi) $a^2 + 0.4a + 0.04$

4. Factorize:

(i) $b^4 - 4b^2c^2 + 4c^4$ (ii) $\frac{9}{4}x^4 - 2 + \frac{4}{9x^4}$

(iii) $2a^3b^3 - 16a^2b^4 + 32ab^5$ (iv) $9(p+q)^2 - 6(p+q)r^2 + r^4$

(v) $x^2y^2 - 0.1xy + 0.0025$ (vi) $(a-b)^2 - 18(a-b) + 81$

5. Factorize:

(i) $4a^2 - 9b^2$ (ii) $16x^2 - 25y^2$ (iii) $100x^2z^2 - y^4$

(iv) $\frac{1}{100}x^4 - 100y^4$ (v) $\frac{64}{81}f^2 - \frac{81}{64}g^4$ (vi) $\frac{x^4}{121} - 121y^2$

6. Factorize:

(i) $(2x + z)^2 - (2x - z)^2$ (ii) $(4a - 9b)^2 - (2a + 5b)^2$

(iii) $169x^4 - (3t + 4)^2$ (iv) $(9x^2 - 4y^2)^2 - (4x^2 - y^2)^2$

(v) $\left(a^2 + 2 + \frac{1}{a^2}\right) - \left(b^2 - 2 + \frac{1}{b^2}\right)$ (vi) $9x^2 + \frac{1}{9x^2} - 4y^2 - \frac{1}{4y^2} + 4$

7. Factorize:

(i) $(x^2 + 2xy + y^2) - 9z^4$ (ii) $(4a^2 + 8ab^2 + 4b^4) - 9c^2$

(iii) $16d^4 - (c^4 - 2c^2d + d^2)$ (iv) $4(x^2 + 2xy^2 + y^4) - 9y^6$

(v) $x^2 - y^2 - 4x - 2y + 3$ (vi) $4x^2 - y^2 - 2y - 1$

8. **Factorize:**

(i) $(\sqrt{ab})^2 - (\sqrt{c})^2$ (ii) $(\sqrt{4x})^2 - (\sqrt{9y})^2$

(iii) $(\sqrt{yz})^2 - \left(\frac{1}{\sqrt{yz}}\right)^2$ (iv) $xzt - \frac{1}{t}$

4.1.2 Factorize the expression of following types.

Type I: $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

This type includes those algebraic expressions which are neither perfect squares nor in the form of the difference of two squares. Factorization of this type is explainedin the following examples.

Example 01 Factorize: $a^4 + a^2b^2 + b^4$.

Solution: $a^4 + a^2b^2 + b^4$

$= (a^4 + b^4) + a^2b^2$ (Rearrange the terms)

$= (a^4 + 2a^2b^2 + b^4) - 2a^2b^2 + a^2b^2$ [by adding and subtracting $2a^2b^2$]

$= \{(a^2)^2 + 2(a^2)(b^2) + (b^2)^2\} - a^2b^2$

$= (a^2 + b^2)^2 - (ab)^2$

$= \{(a^2 + b^2) - ab\}\{(a^2 + b^2) + ab\}$ $[\because a^2 - b^2 = (a - b)(a + b)]$

$= (a^2 - ab + b^2)(a^2 + ab + b^2)$

Example 02 Factorize: $a^4 + 4b^4$

Solution: $a^4 + 4b^4$

$= (a^2)^2 + (2b^2)^2$

$= (a^2)^2 + (2b^2)^2 + 2(a^2)(2b^2) - 2(a^2)(2b^2)$

$= \{(a^2)^2 + 2(a^2)(2b^2) + (2b^2)^2\} - 4a^2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2$

$= \{(a^2 + 2b^2) - 2ab\}\{(a^2 + 2b^2) + 2ab\}$ $[\because a^2 - b^2 = (a - b)(a + b)]$

$= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$

[For completing the square adding and subtracting $2(a^2)(2b^2)$]

Example 03 Factorize: x^8+x^4+1

Solution: x^8+x^4+1

$= (x^8+1)+x^4$

$= \{(x^4)^2+(1)^2+2(x^4)(1)\}-2(x^4)(1)+x^4$ [For completing the square adding and subtracting $2(x^4)(1)$]

$= (x^4+1)^2-x^4$

$= (x^4+1)^2-(x^2)^2$

$= \{(x^4+1)-x^2\}\{(x^4+1)+x^2\}$ $[\because a^2-b^2=(a-b)(a+b)]$

$= \{(x^4+x^2+1)\}(x^4-x^2+1)$

$= \{(x^2+1)^2-2x^2+x^2\}(x^4-x^2+1)$

$= \{(x^2+1)^2-x^2\}(x^4-x^2+1)$

$= (x^2+x+1)(x^2-x+1)(x^4-x^2+1)$

Type II: x^2+px+q

This type of expression can be factorized by breaking the middle term process.

Example 01 Factorize $y^2+7y+12$

Solution: $y^2+7y+12$

$= y^2+3y+4y+12$

$= y(y+3)+4(y+3)$

$= (y+3)(y+4)$

Example 02 Factorize $x^2+13xy-30y^2$

Solution: $x^2+13xy-30y^2$

$= x^2+15xy-2xy-30y^2$

$= x(x+15)-2y(x+15)$

$= (x+15)(x-2y)$

Type III: $ax^2+bx+c, a\neq 0.$

To factorize the expression ax^2+bx+c, $a\neq 0$, the following steps are needed:

(i) Find the product ac, where a is coefficient of x^2 and c is constant.

(ii) Find two numbers x_1 and x_2 such that $x_1+x_2=b$ and $x_1x_2=ac$.

To explain this method the following examples are helpful.

Example 01 Factorize $10x^2-19xy+6y^2$

Solution: $10x^2-19xy+6y^2$

$= 10x^2-15xy-4xy+6y^2$

$= 5x(2x-3y)-2y(2x-3y)$

$= (2x-3y)(5x-2y)$

Example 02 Factorize $4x^2+12x+5$

Solution: $4x^2+12x+5$

$= 4x^2+10x+2x+5$

$= 2x(2x+5)+1(2x+5)$

$= (2x+5)(2x+1)$

Example 03 Factorize $3x^2+4x-4$

Solution: $3x^2+4x-4$

$= 3x^2+6x-2x-4$

$= 3x(x+2)-2(x+2)$

$= (x+2)(3x-2)$

Example 04 Factorize $6x^2-x-7$

Solution: $6x^2-x-7$

$= 6x^2-7x+6x-7$

$= x(6x-7)+1(6x-7)$

$= (6x-7)(x+1)$

Exercise 4.2

1. Factorize the following:

(i) $a^4+a^2x^2+x^4$ (ii) b^4+b^2+1

(iii) $a^8+a^4x^4+x^8$ (iv) z^8+z^4+1

2. Factorize:

(i) x^4+4y^4 (ii) $36x^4z^4+9y^4$

(iii) $4t^4+625$ (iv) $4t^4+1$

3. Resolve into factors:

(i) $x^2+3x-10$ (ii) $a^2b^2-3ab-10$

(iii) $y^2+7y-98$ (iv) $x^2y^2z^2+2xyz-24$

4. Resolve into factors:

(i) $9y^2+21yz-8z^2$ (ii) $42x^2-8x-2$

(iii) $4x^2+12x+5$ (iv) $3x^2-38xy-13y^2$

Type IV: $(ax^2 + bx + c)(ax^2 + bx + d)+k$

$(x+a)(x+b)(x+c)(x+d)+k$

$(x+a)(x+b)(x+c)(x+d)+kx^2$

We shall explain the procedure of factorizing of these types expressions with the help of following examples.

Example 01 Factorize: $(x^2+5x+4)(x^2+5x+6)-120$

Solution: $(x^2+5x+4)(x^2+5x+6)-120$

Let $x^2+5x = t$, then we have

$(t+4)(t+6) -120$

$= t^2+10t+24-120$

$= t^2+10t-96$

$= t^2-6t+16t-96$ (by factorizing)

$= t(t-6)+16(t-6)$

$= (t-6)(t+16)$

$= (x^2+5x-6)(x^2+5x+16)$ $[\because t=x^2+5x]$

$= (x^2-x+6x-6)(x^2+5x+16)$

$= [x(x-1)+6(x-1)](x^2+5x+16)$

$= (x-1)(x+6)(x^2+5x+16)$

Example 02 Factorize: $(x+1)(x+2)(x+3)(x+4)-15$

Solution: $(x+1)(x+2)(x+3)(x+4)-15$

Here 1+4= 2+3 =5

$(x+1)(x+4)(x+2)(x+3)-15$ by arranging the factors

$= (x+1)(x+4)(x+2)(x+3)-15$

$= (x^2+5x+4)(x^2+5x+6)-15$

$= (t+4)(t+6)-15$ where $t = x^2+5x$

$= t^2+10t+24-15$

$= t^2+10t+9$

$= (t+1)(t+9)$

$= (x^2+5x+1)(x^2+5x+9)$ $\because t = x^2+5x$

Example 03 Factorize: $(x+2)(x-2)(x-3)(x+3)+(-2x^2)$

Solution: $(x+2)(x-2)(x-3)(x+3)+(-2x^2)$

$= (x+2)(x-2)(x-3)(x+3)+(-2x^2)$

$= (x^2-2^2)(x^2-3^2)-2x^2$ $[\because (a+b)(a-b)=a^2-b^2)]$

$= (x^2-4)(x^2-9)-2x^2$

$=x^4-9x^2-4x^2+36-2x^2$

$=x^4-15x^2+36$

$=x^4-3x^2-12x^2+36$

$= x^2(x^2-3)-12(x^2-3)$

$= (x^2-3)(x^2-12)$

$= [(x)^2-(\sqrt{3})^2][(x)^2-(2\sqrt{3})^2]$

$= (x-\sqrt{3})(x+\sqrt{3})(x-2\sqrt{3})(x+2\sqrt{3})$

Exercise 4.3

1. Factorize the following:

(i) $(x^2-4x-5)(x^2-4x-12)-144$
(ii) $(x^2+5x+6)(x^2+5x+4)-3$
(iii) $(x^2-2x+3)(x^2-2x+4)-42$
(iv) $(x^2-8x+4)(x^2-8x-4)+15$
(v) $(x^2+9x-1)(x^2+9x+5)-7$
(vi) $(x^2-5x+4)(x^2-5x+6)-120$

2. Factorize:

(i) $(x+1)(x+2)(x+3)(x+4)-48$
(ii) $(x+2)(x+3)(x+4)(x+5)-24$
(iii) $(x-1)(x-2)(x-3)(x-4)-99$
(iv) $(x-3)(x-5)(x-7)(x-9)+15$
(v) $(x-1)(x-2)(x-3)(x-4)-224$
(vi) $(x-2)(x-3)(x-4)(x-5)-255$

3. Factorize

(i) $(x-2)(x-3)(x+2)(x+3)-2x^2$

(ii) $(x-1)(x+1)(x+3)(x-3)-3x^2-23$

(iii) $(x-1)(x+1)(x-3)(x+3)+4x^2$

(iv) $(x-2)(x+2)(x-4)(x+4)-14x^2$

(v) $(x+5)(x+2)(x-5)(x-2)+4x^2$

(vi) $(x^2-x-12)(x^2-x-12)-x^2$

Type V: $a^3+3a^2b+3ab^2+b^3$ **and** $a^3-3a^2b+3ab^2-b^3$

As we know that

$$a^3+3a^2b+3ab^2+b^3=(a+b)^3$$

and $$a^3-3a^2b+3ab^2-b^3=(a-b)^3.$$

The following examples will help us to understand the factorization of the types mentioned above.

Example 01 Factorize (i) $8x^3+12x^2y+6xy^2+y^3$ (ii) $64x^3-12x^2+\frac{3x}{4}-\frac{1}{64}$

Solution (i): $8x^3+12x^2y+6xy^2+y^3$

$=(2x)^3+3(2x)^2(y)+3(2x)(y)^2+(y)^3$ $\quad [\because a^3+3a^2b+3ab^2+b^3=(a+b)^3]$

$=(2x+y)^3$

Solution (ii): $64x^3-12x^2+\frac{3x}{4}-\frac{1}{64}$

$=(4x)^3-3(4x)^2(\frac{1}{4})+3(4x)\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^3$ $\quad [\because a^3-3a^2b+3ab^2-b^3=(a-b)^3]$

$=\left(4x-\frac{1}{4}\right)^3$

Exercise 4.4

1. Factorize the following:

(i) $b^3+3b^2c+3bc^2+c^3$

(ii) $8x^3+12x^2y+6xy^2+y^3$

(iii) $64x^3+12x^2+\frac{3x}{4}+\frac{1}{64}$

(iv) $8x^3+36x^2+54x+27$

(v) $\frac{1}{27}+\frac{1}{3}y^2+y^4+y^6$

(vi) $\frac{8}{27}x^3+2x^2y+\frac{9}{2}xy+\frac{27}{8}y^3$

(vii) $\frac{64}{27}+\frac{16}{3}x+4x^2+x^3$

(viii) $\frac{z^3}{8}+\frac{z^2y}{4}+\frac{zy^2}{6}+\frac{y^3}{27}$

2. Factorize:

(i) $d^3-6d^2c+12dc^2-8c^3$

(ii) $x^6+\frac{16}{3}x^2-4x^4-\frac{64}{27}$

(iii) $\frac{x^3}{125}-\frac{3}{25}x^2y+\frac{3}{5}xy^2-y^3$

(iv) $125z^3-75z^2y^2+15zy^4-y^6$

(v) $\frac{z^3}{27}-2z^2y+36zy^2-216y^3$

(vi) $\frac{b^6}{27}-\frac{b^4c^2}{6}+\frac{b^2c^4}{4}-\frac{c^6}{8}$

(vii) $216+\frac{9z^2}{2}-54z-\frac{z^3}{8}$

(viii) $\frac{8}{27}x^3-2x^2y+\frac{9}{2}xy^2-\frac{27}{8}y^3$

Type VI: $a^3\pm b^3$

As we know that,

$$a^3+b^3=(a+b)(a^2-ab+b^2),$$

and $$a^3-b^3=(a-b)(a^2+ab+b^2).$$

The following example will help us to understand the factorization of above mentioned type.

Example 01 Factorize $8x^3+27$

Solution: $8x^3+27$

$= (2x)^3+(3)^3$

$= (2x+3)[(2x)^2-(2x)(3)+(3)^2]$ $\quad [\because a^3+b^3=(a+b)(a^2-ab+b^2)]$

$= (2x+3)(4x^2-6x+9)$

Therefore $8x^3+27=(2x+3)(4x^2-6x+9)$

Example 02 factors of $108x^4 - 256xz^3$.

Solution: $108x^4 - 256xz^3$

$= 4x(27x^3 - 64z^3)$

$= 4x[(3x)^3 - (4z)^3$ $\quad [\because a^3 - b^3 = (a-b)(a^2+ab+b^2)]$

$= 4x(3x-4z)[(3x)^2+(3x)(4z)+(4z)^2]$

$= 4x(3x-4z)(9x^2+12xz+16z^2)$

Therefore: $108x^3 - 256xz^3 = 4x(3x-4z)\ (9x^2+12xz+16z^2)$.

> **Note:** For complete factorization of the expression of type $x^n - y^n$, where $n = 6k, \forall k \in N$; apply difference of two squares formula first, then apply difference of two cubes formula.

Exercise 4.5

1. Factorize the following:

(i) $x^3 + 8y^3$ (ii) $a^{11} + a^2b^9$ (iii) $a^6 + 1$ (iv) $a^3b^3 + 512$

(v) $a^3b^3 + 27b^6$ (vi) $\dfrac{x^3}{125} + \dfrac{125}{x^3}$ (vii) $x^9 + x^3y^6z^9$ (viii) $\dfrac{x^6}{27} + \dfrac{8}{x^3}$

2. Factorize

(i) $x^3 - 8y^3$ (ii) $x^9 - 8y^9$ (iii) $1000 - \dfrac{x^3y^3}{125}$ (iv) $a^6 - b^6$

(v) $\dfrac{x^6}{64} - \dfrac{64}{x^{12}}$ (vi) $x^{12} - y^{12}$ (vii) $\dfrac{27}{x^3} - 8y^6$ (viii) $8x^6 - \dfrac{1}{729}$

4.2 Remainder and Factor Theorems

Remainder and factor theorems are usually used to find the factors of the polynomial expressions of third and higher degrees of the polynomials.

4.2.1 State and prove Remainder Theorem and Explain through examples

Statement:

When a polynomial $p(x)$ of degree $n \geq 1$ is divided by a linear polynominal $(x-a)$, then remainder R is obtained by putting $x=a$ i.e $R = p(a)$. We can write $p(x)$ as $p(x)=q(x)(x-a)+R$, (which is called division algorithm) where R is a constant (remainder), and the degree of $q(x)$ is less than the degree of $p(x)$ by 1.

Proof:

By division algorithm

$p(x) = q(x)(x-a)+R$ .... (i)

As (i) is an identity, so it is true for all values of x.

$\therefore$ Putting $x = a$ in (i), we get

$p(a) = q(a)\times(a-a)+R,$

$p(a) = q(a)\times 0+R$

$\Rightarrow$ $p(a) = R$ = remainder. Hence Proved.

4.2.2 Find remainder (without dividing) when a polynomial is divided by a linear polynomial.

The following examples help us to use of remainder theorem.

Example 01 Find the remainder when x^2-3x+4 is divided by $x-2$

Solution: Let $p(x)=x^2-3x+4$

Here a=2 by remainder theorem

$p(2) = (2)^2 - 3(2) + 4$

$= 4 - 6 + 4 = 8 - 6 = 2$

$p(2) = R = 2$

Thus, the reminder is 2.

Example 02 Find the value of k, if the polynomial x^3+kx^2+3x-4 leaves a remainder -2 when divided by $x+2$.

Solution:

Here $p(x) = x^3+kx^2+3x-4$

$\therefore$ $p(-2) = (-2)^3 + k(-2)^2 + 3(-2) - 4$

$\Rightarrow$ $-2 = 4k - 18$ $[\because \text{Remainder} = -2 = p(-2)]$

$\Rightarrow$ $4k = -2 + 18$

$\Rightarrow$ $4k = 16$

$\Rightarrow$ $k = 4,$

Thus, the value of k is 4.

4.2.3 Zero of a polynomial.

Let $p(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ be a polynomial with real coefficient. By putting $x=a$ in the polynomial $p(x)$, if the value of $p(x)$ becomes zero i.e. $p(a)=0$. Then 'a' is called the zero of polynomial $p(x)$.

Example 01 If $p(x) = x+7$ then -7 is a zero of polynomial as $p(-7)=-7+7=0$.

4.2.4 State and prove factor theorem.

Statement:

The linear polynomial $x–a$ is a factor of the polynomial $p(x)$ iff, $p(a) = 0$.

Proof:

Let $q(x)$ be the quotient and R be the remainder when a polynomial $p(x)$ is divided by $x–a$

Then by division algorithm, we have

$p(x)=(x-a)q(x)+R$ .... (i)

By the Remainder theorem,

$R = p(a)$, using in (i), we get

Thus, $\boxed{p(x) = q(x)(x-a)+p(a)}$

It is given that $p(a)=0$, then $p(x)=q(x)(x-a)$.

Note that $p(x)$ is expressed as a product of $q(x)$ and $(x - a)$.

Thus, $(x-a)$ is a factor of the polynomial $p(x)$. Hence proved.

The following examples will help us to use of factor theorem.

Example 01 Determine whether $x + 2$ is a factor of $x^3 + \frac{9}{2}x^2 + 3x - 4$ or not.

Solution: Let $p(x) = x^3 + \frac{9x^2}{2} + 3x - 4$

putting $x = -2$, we get

$\therefore R = p(-2) = (-2)^3 + \frac{9}{2}(-2)^2 + 3(-2) - 4$

$= -8 + 18 - 6 - 4$

$= -18 + 18 = 0 \Rightarrow R = 0,$

Remainder = 0,

$\therefore$ $x+2$ is factor of $p(x)$

Example 02 Determine whether $x + 3$ is a factor of Thus, $x^3 - x^2 - 8x + 12$

Solution:

Let $p(x) = x^3 - x^2 - 8x + 12$

$\therefore$ putting $x = -3$, we get

$R = p(-3) = (-3)^3 - (-3)^2 - 8(-3) + 12$

$= -27 - 9 + 24 + 12$

$R = p(-3) = -36 + 36 = 0$

As remainder is found to be zero

$\therefore$ $x+3$ is factor of $x^3 - x^2 - 8x + 12$

Exercise 4.6

1. Find the remainder by using the remainder theorem when

(i) $x^3 - 6x^2 + 11x - 8$ is divided by $(x-1)$

(ii) $x^3 + 6x^2 + 11x + 8$ is divided by $(x+1)$

(iii) $x^3 - x^2 + 14$ is divided by $(x-2)$

(iv) $x^3 - 3x^2 + 4x - 14$ is divided by $(x+2)$

(v) $(2y-1)^3 + 6(3+4y) - 9$ is divided by $(2y+1)$

(vi) $4y^3 - 4y + 3$ is divided by $(2y-1)$

(vii) $(2y+1)^3 - 6(3-4y) - 10$ is divided by $(2y-1)$

(viii) $x^4 + x^2y^2 + y^4$ is divided by $(x-y)$.

2. Find the value of m, if $p(y) = my^3 + 4y^2 + 3y - 4$ and $q(y) = y^3 - 4y + m$ leaves the same remainder when divided by $(y - 3)$.

3. If the polynomial $4x^3 - 7x^2 + 6x - 3k$ is exactly divisible by $(x + 2)$, find the value of k.

4. Find the value of r, if $(y+2)$ is a factor of the polynomial $3y^2 - 4ry - 4r^2$.

4.3 Synthetic Division

Synthetic division is a method to divide a polynomial by a linear polynomial.

4.3.1 Describe the method of synthetic division.

The method of synthetic division is described with the help of following example.

Example 1. Divide the polynomial $p(x) = x^3 - 3x^2 + 5x + 7$ *by* $(x-1)$ using synthetic division

Solution: Here $x - 1 = 0 \Rightarrow x = 1$, (1 is a multiplier).

Write down the coefficients of the given polynomial.

Thus,

1	1	−3	5	7	(Row 1)
		1	−2	3	(Row 2)
	1	−2	3	10 = R	(Row 3)

Description

Step I. In Row 1, write the coefficients and constant term of the given polynomial $p(x)$ in the descending order. If any term is missing in $p(x)$ then insert zero for that term.

Step II. Write the first coefficient in Row 3, below its position in Row 1.

Step III. Write the product of 1 (the multiplier) and the coefficient (1) in the Row 3 beneath the 2nd coefficient in Row 2, and add, putting the sum below them in the Row 3 and so on.

Thus, $q(x)=x^2-2x+3$ and $p(1) = R = 10$.

> Note: Degree of $q(x)$=[Degree of $p(x)$]$-1= 3-1=2$
> and the last element of the row 3, is the remainder.

4.3.2 Use of Synthetic division to:

(a) Find quotient and remainder when a given polynomial is divided by a linear polynomial.

(b) Find the value(s) of unknown(s) if zeros of the polynomial are given.

(c) Find the value(s) of unknown(s) if the factors of the polynomial are given.

Example 01 Find quotient and remainder when,
$p(x) = x^4-12x^3+50x^2-84x+49$ is divided by a linear polynomial $x-5$

Solution: Given that $p(x)=x^4-12x^3+50x^2-84x+49$,
and linear polynomial $x-a = x-5$, i.e., $a= 5$ is the multiplier = 5
$p(5) = R = ?$ and $q(x) = ?$
To find quotient and the remainder, we will use synthetic division method as under.

5	1	−12	50	−84	49	(Row 1)
		5	−35	75	−45	(Row 2)
	1	−7	15	−9	4 = R	(Row 3)

Thus, $q(x)=x^3-7x^2+15x-9$ and $R=4$ are the required quotient and remainder respectively.

Note: $R \neq 0$, therefore $(x-5)$ is not the factor of given polynomial $p(x)$.

Example 02 For what value of m, 1 is a zero of the polynomial $p(x)= x^3-mx^2+x-1$?

Solution: Given that,

$p(x)= x^3-mx^2+x-1$

Here, the multiplier $a= 1$.

By synthetic division method we have,

1	1	$-m$	1	-1	(Row 1)
		1	$1-m$	$2-m$	(Row 2)
	1	$1-m$	$2-m$	$1-m=R$	(Row 3)

Since 1 is the zero of $p(x)$, so $R=0$

$\Rightarrow \quad 1-m=0$

$\Rightarrow \quad m=1$

Thus, for 1 as a zero of the polynomial $p(x)$ m must be equal to 1.

Exercise 4.7

1. **By using synthetic division method to divide the following polynomials and also find their quotient and remainder.**

(i) $p(x)=x^3-x^2+x-1$ by $x-1$

(ii) $p(x)=x^3-x^2-x-1$ by $x+1$

(iii) $p(x)=x^3-6x^2+11x-6$ by $x+2$

(iv) $p(x)=x^3+6x^2-11x-6$ by $x-2$

(v) $p(x)=x^4-x^3+x^2-x-1$ by $x+2$

(vi) $p(x)=x^4+x^3-x^2+x-1$ by $x-1$

(vii) $p(x)=x^5+x^3-2x^2-3$ by $x+3$

(viii) $p(x)=x^5-x^4+x^3-3x^2+6x-6$ by $x-3$

(ix) $p(x)=2x^4-2x^3+100x^2-168x+95$ by $x-2$

(x) $p(x)=6x^4-72x^3+300x^2-564x+270$ by $x-5$

2. For what value of k, -2 is zero of the polynomial $p(x)=x^3+x^2-14x-k$
3. For what value of m, $(x-2)$ be factor of $x^3+mx^2-7x-10$
4. For what value of m, $(x+2)$ is factor of $4x^3-7x^2+6x-3m$
5. For what value of m, -1 is a zero of the polynomial, $p(x)=2x^3-4mx^2+x-1$

4.4 Factorization of Cubic Polynomial

We have already studied the method of solving linear and quadratic polynomials. Now we will find the factors of cubic polynomials using factor theorem.

4.4.1 Use factor theorem to factorize a cubic polynomial

To factorize a cubic polynomial by factor theorem, it is necessary that one of the factor or more of the zeros of the polynomial is (are) known.

Let us see the following examples:

Example 01 Factorize $x^3-6x^2+11x-6$

Solution: Let $p(x)= x^3-6x^2+11x-6$

The factors of 6 are $\pm1,\pm2,\pm3,\pm6$

$p(1)=(1)^3-6(1)^2+11(1)-6$

$p(1)=1-6+11-6$

$p(1)=0$

Hence, $x-1$ is a factor of $p(x)$

By synthetic division

1	1	−6	11	−6	(Row 1)
		1	−5	6	(Row 2)
	1	−5	6	0	(Row 3)

By division algorithm, $p(x) = (x-a)\, q(x)+R$

$\therefore\quad p(x)=(x-1)(x^2-5x+6)+0$

$=(x-1)\{x^2-2x-3x+6\}$

$=(x-1)\{x(x-2)-3(x-2)\}$

$p(x)=(x-1)(x-2)(x-3)$

Example 02 Factorize x^3-4x^2+x+6

Solution: The factors of 6 are $\pm1,\pm2,\pm3,\pm6$

If $x-1$ is a factor of $p(x)$

Then,

$p(1)=(1)^3-4(1)^2+1+6$

$=1-4+1+6$

$=4\neq0$

Hence, $x-1$ is not factor of $p(x)$.
If $x+1$ is a factor of $p(x)$
Then,

$$p(-1)=(-1)^3-4(-1)^2+1(-1)+6$$
$$=-1-4-1+6$$
$$=0$$

Hence, $x-1$ is factor of $p(x)$.
By synthetic division

-1	1	-4	1	6	(Row 1)
		-1	5	-6	(Row 2)
	1	-5	6	0	(Row 3)

By division algorithm, $p(x) = (x\text{-}a)\,q(x) + R$

$$\therefore \quad p(x)=(x+1)(x^2-5x+6)+0$$
$$=(x+1)\{x^2-2x-3x+6\}$$
$$=(x+1)\{x(x-2)-3(x-2)\}$$
$$p(x)=(x+1)(x-2)(x-3)$$

Exercise 4.8

Factorize by using factor theorem

1. x^3-x^2+x-1 **2.** x^3+x^2-x-1 **3.** $x^3-6x^2+11x-6$

4. $x^3+5x^2-4x-20$ **5.** $x^3-2x^2+9x-18$ **6.** $6x^3+7x^2-x-2$

7. $x^3+8x^2+19x+12$ **8.** $2x^3+9x^2+10x+3$ **9.** $x^3+12x^2+44x+48$

Review Exercise 4

1. True and false questions

Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) $x^2+x-6=(x+3)(x-2)$ T/F

(ii) $a^3+27=(a+3)(a^2-3a+9)$ T/F

(iii) $b^3-8=(b-2)(b^2+2b+4)$ T/F

(iv) $a^4 - b^4 = (a-b)(a+b)(a+b)^2$ T/F

(v) $a^6+b^6=(a^3+b^3)(a^3-b^3)$ T/F

(vi) $a^5+b^5 = (a+b)^5$ T/F

2. Complete the following sentences

(i) $16x^2-y^4=(4x-y^2)$ ____________

(ii) $x^3-64y^3=(x-4y)$ ____________

(iii) $x^2+5x+6=(x+2)$ ____________

(iv) $x^2+y^2=(x-y)^2$ ____________

(v) $a^3+27b^3=(a+3b)$ ____________

3. Tick (✓) the correct answers

(i) Factors of $a^2+2a-24$ are:

(a) $a+4, a-6$ (b) $a-4, a+6$

(c) $a+3, a-8$ (d) $a+8, a-3$

(ii) Factors of $a^2+2ab+b^2-c^2$ is:

(a) $(a-b+c)(x-b-c)$ (b) $(a+b+c)(a-b-c)$

(c) $(a+b+c)(a+b-c)$ (d) $(a+b+c)(a-b-c)$

(iii) Factors of x^3+y^3 is:

(a) $(x-y)(x^2+xy+y^2)$ (b) $(x+y)(x^2+xy+y^2)$

(c) $(x+y)(x^2+xy-y^2)$ (d) $(x+y)(x^2-xy+y^2)$

(iv) Factors of y^3-27z^3 are:

(a) $y-3z, y^2+3yz+9z^2$ (b) $y-3z, y+3z+9z^2$

(c) $y-3z, y^2-3yz+9z^2$ (d) $y+3z, y^2-3yz+9z^2$

(v) In simplified form $\frac{1}{x+y}+\frac{y}{x^2-y^2}=$

(a) $\frac{y+1}{x^2-y^2}$ (b) $\frac{x}{x^2-y^2}$

(c) $\frac{y}{x^2-y^2}$ (d) $\frac{y-1}{x^2-y^2}$

(vi) Find m, so that x^2+4x+m is a complete square

(a) 16 (b) -16

(c) 4 (d) -4

Summary

- Factorization is the process in which we express the given polynomial (expression) as a product of two or more expressions.
- We learn and resolve into factors of the following types of formulas:

  (i) $ka + kb + kc = k(a + b + c)$

  (ii) $\underline{ac + ad} + \underline{bc + bd} = a(c + d) + b(c + d) = (a + b)(c + d)$

  (iii) $a^2 + 2ab + b^2 = (a)^2 + 2(a)(b) + (b)^2 = (a + b)^2$

  (iv) $a^2 - 2ab + b^2 = (a)^2 - 2(a)(b) + (b)^2 = (a - b)^2$

  (v) $a^2 - b^2 = (a-b)(a + b)$
- Remainder and factor theorems are two important theorems; these are used to find the factors of such type of polynomials which cannot be solved by the given formulas:

  (i) $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$ (ii) $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$

  (iii) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ (iv) $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
- **Zeros of the Polynomial**

  If specific number $x=a$ is substituted for the variable 'x' in a polynomial $p(x)$ such that, the value of $p(a)$ is zero, then 'a' is called a zero of the Polynomial $p(x)$.
- **Factor theorem** can also be stated as:

  The linear polynomial $(x - a)$ is a factor of the polynomial $p(x)$ iff, $p(a)=0$

- **Description of synthetic division method**

  Step I. Write in Row 1, the coefficients of $p(x)$ in the descending powers of x. If any term is missing in $p(x)$, then insert zero for that term.

  Step II. Write the first coefficient in Row below its position in Row 1.
- **Step III.** Write the product of 2 multiplier and this coefficient in the Row 2 beneath the 2nd coefficient in Row 1, and added, putting the sum below them in the Row 3 and so on.

Unit 5

• Weightage = 5%

ALGEBRAIC MANIPULATION

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Find Highest Common Factor (H.C.F) and Least Common Multiple (L.C.M) of algebraic expressions by factorization method.
- Know the relationship between H.C.F and L.C.M
- Use division method to determine highest common factor and least common multiple.
- Solve real life problems related to HCF and LCM
- Use highest common factor and least common multiple to reduce fractional expressions involving addition, subtraction, multiplication and division.
- Find square root of an algebraic expression by factorization and division methods.
- Solve real life problems related to HCF and LCM.

Introduction:

Algebraic manipulation refers to manipulation of algebraic expressions, often into a simpler form or form which is more easily handled and dealt with. It is one of the most basic, necessary and important skills in a problem solving of algebraic expression.

In this unit, we will discuss HCF, LCM and square root of the algebraic expressions by both factorization and division methods and their applications in daily life.

5.1 Highest Common Factor (HCF) / Greatest Common Division and Least Common Multiple (LCM)

5.1.1 Find Highest Common Factor (HCF) and Least Common Multiple (LCM) of Algebraic expression by Factorization method.

(a) Highest Common Factor (HCF) by Factorization method

For finding the HCF of the given expression, first we find the factors of each polynomial. Then we take the product of their common factors. This product of common factors is known as HCF by factorization.

Notes: 1. In case there is no common factor then HCF is 1.
2. HCF is also called GCD (Greatest Common Divisor).

Example 01 Find the HCF of the following expression by using factorization method.

(i) $x^2 + x - 20$ and $x^2 + 12x + 35$

(ii) $(x + 1)^2$, $x^2 - 1$ and $x^2 + 4x + 3$

Solution (i): We factorize the given expression $x^2 + x - 20$ and $x^2 + 12x + 35$

The factors are as under:

$$x^2 + x - 20 = x^2 + 5x - 4x - 20$$
$$= x(x + 5) - 4(x + 5)$$
$$= (x + 5)(x - 4)$$

and
$$x^2 + 12x + 35 = x^2 + 7x + 5x + 35$$
$$= x(x + 7) + 5(x + 7)$$
$$= (x + 5)(x + 7)$$

Common factor in both the expression is $(x + 5)$

$\therefore$ HCF $= x + 5$

Solution (ii): We factorize the given expression $(x+1)^2$, x^2-1 and x^2+4x+3

The factors are as under:

$$(x+1)^2 = (x+1)(x+1)$$
$$x^2-1 = (x+1)(x-1)$$

and $x^2+4x+3 = x^2+3x+x+3$
$$= x(x+3)+1(x+3)$$
$$= (x+3)(x+1)$$

Common factor in all the three expression is $(x+1)$

$\therefore$ H.C.F $= x+1$.

Example 02 Find the H.C.F of the following expression by using factorization method.

a^3-b^3, a^6-b^6

Solution :

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$
$$a^6-b^6 = (a^2)^3-(b^2)^3$$
$$= (a^2-b^2)\{(a^2)^2+a^2b^2+(b^2)^2\}$$
$$= (a-b)(a+b)\{(a^2)^2+2a^2b^2+(b^2)^2-a^2b^2\}$$
$$= (a-b)(a+b)\{(a^2+b^2)^2-(ab)^2\}$$
$$= (a-b)(a+b)(a^2+b^2-ab)(a^2+b^2+ab)$$

HCF $= (a-b)(a^2+b^2+ab) = a^3-b^3$

(b) Least Common Multiple (LCM) by Factorization method

Least common multiple (LCM) of two or more polynomials is the expression of least degree which is divisible by the given polynomials. To find LCM by Factorization we use the following formula:

LCM = Common factors × non common factors

Example 01 Find the LCM of x^3-8 and x^2+x-6

Solution: Now find the factors of x^3-8 and x^2+x-6

$\therefore$ $x^3-8 = (x)^3-(2)^3 = (x-2)(x^2+2x+4)$

and $x^2+x-6 = x^2+3x-2x-6 = x(x+3)-2(x+3)$
$$= (x+3)(x-2)$$

Common factor $= (x-2)$

Non common factors $= (x+3)(x^2+2x+4)$

$\because$ LCM= Common factors × non common factors

$\therefore$ LCM$=(x-2)(x^2+2x+4)(x+3)=(x^3-8)(x+3)$

Example 02 Find the L.C.M of x^3-1, x^3-2x^2+x

Solution: Now find the factors of x^3-1 and x^3-2x^2+x

$\therefore$ $x^3-1=(x)^3-(1)^3=(x-1)(x^2+x+1)$

and $x^3-2x^2+x=x(x^2-2x+1)=x(x-1)^2$

$\because$ LCM= Common factors × non common factors

$\therefore$ $\text{LCM}=x(x-1)^2(x^2+x+1)$.

5.1.2 Know the Relationship between HCF and LCM

The relation between HCF and LCM of two polynomials $p(x)$ and $q(x)$ is expressed as under

$$\text{HCF} \times \text{LCM} = p(x) \times q(x)$$

Example 01 Find the HCF and LCM of the polynomials $p(x)$ and $q(x)$ given below, and verify the relation of HCF and LCM.

$p(x) = x^2-5x+6$ and $q(x) = x^2-9$.

Solution: First factorize the polynomials $p(x)$ and $q(x)$ into irreducible factors,

We have,

$p(x) = x^2-5x+6$

$= x^2-3x-2x+6$

$= x(x-3)-2(x-3)$

$= (x-3)(x-2)$

and $q(x) = x^2-9 = (x-3)(x+3)$

Thus, H.C.F $= (x-3)$

and L.C.M $= (x-2)(x-3)(x+3) = (x-2)(x^2-9)$

Now, find the product of $p(x)$ and $q(x)$.

so, $p(x) \times q(x) = (x^2-5x+6) \times (x^2-9)$ . . . (i)

LCM × HCF $=(x-2)(x-3)(x+3) \times (x-3)$

$\Rightarrow$ $= (x^2-5x+6) \times (x^2-9)$ . . . (ii)

From results (i) and (ii), we have

LCM × HCF $= p(x) \times q(x)$, Hence, verified.

Example 02 Find the LCM of the following polynomials by using the formula.

$p(x) = x^2 + 14x + 48$ and $q(x) = x^2 + 8x + 12$.

Solution: Now first we have to find the HCF of the $p(x)$ and $q(x)$.

$$p(x) = x^2 + 14x + 48$$
$$= x^2 + 8x + 6x + 48$$
$$= x(x+8) + 6(x+8)$$
$$= (x+6)(x+8)$$

and $q(x) = x^2 + 8x + 12$
$$= x^2 + 6x + 2x + 12$$
$$= x(x+6) + 2(x+6)$$
$$= (x+2)(x+6)$$

so, HCF of $p(x)$ and $q(x)$ is $(x + 6)$.

$\because$ $$\text{LCM} = \frac{p(x) \times q(x)}{\text{HCF}}$$

$\therefore$ $$\text{LCM} = \frac{(x^2 + 14x + 48) \times (x^2 + 8x + 12)}{(x+6)}$$

$\Rightarrow$ $$\text{LCM} = \frac{(x+6)(x+8)(x+2)(x+6)}{(x+6)}$$

so, $\boxed{\text{LCM} = (x+2)(x+6)(x+8).}$

Notes:

If $p(x)$, $q(x)$ and $r(x)$ are three polynomials having no common factor to them, then

1. LCM would be their product. i.e $\boxed{\text{LCM} = p(x)q(x)r(x)}$

2. HCF would be unity (one). i.e $\boxed{\text{HCF} = 1}$

5.1.3 Use division method to determine highest common factor and least common multiple .

To find the HCF of two or more algebraic expressions by division method, the following examples will help us to understand the method.

Example 01 Find the HCF by division method of the following polynomials:

$2x^3 + 7x^2 + 4x - 4$ and $2x^3 + 9x^2 + 11x + 2$.

Solution: Now, by actual division, we have,

$$\begin{array}{r|l} & \quad 1 \\ 2x^3+7x^2+4x-4 & \;2x^3+9x^2+11x+2 \\ & -2x^3 \pm 7x^2 \pm \;4x \quad 4 \\ \hline & \qquad 2x^2 + \;7x+6 \end{array}$$

Again,

$$\begin{array}{r|l} & \quad x \\ 2x^2+7x+6 & \;2x^3+7x^2+4x-4 \\ & -2x^3 \pm 7x^2 \pm 6x+0 \\ \hline & \qquad\qquad -2x-4 \end{array}$$

We take common factor -2 from $-2x - 4$ and omit it.

$$\begin{array}{r|l} & \;2x+3 \\ x+2 & \;2x^2+7x+6 \\ & -2x^2 \pm 4x+0 \\ \hline & \qquad 3x+6 \\ & \quad -3x \pm 6 \\ \hline & \qquad 0 \quad 0 \end{array}$$

Required HCF is $x + 2$.

Example 02 Find the HCF by division method of the following polynomials:

$x^2 + 2x+1$, $x^2 - 1$ and $x^2+ 4x+ 3$.

Solution: First we find the HCF of any two expression then its HCF with third

Now, by actual division, we have,

$$\begin{array}{r|l} & \quad 1 \\ x^2+2x+1 & x^2+4x+3 \\ & x^2 \pm 2x \pm 1 \\ \hline & \quad 2x+2 \end{array}$$

We take common factor +2 from $2x+2$ and omit it.

Again,

$$\begin{array}{r|l} & x+1 \\ \hline x+1 & x^2+2x+1 \\ & -x^2 \pm x+0 \\ \hline & \quad x+1 \\ & \quad -x \pm 1 \\ \hline & \quad 0 \quad 0 \end{array}$$

$\therefore$The HCF of x^2+2x+1 and x^2+4x+3 is $x+1$

Now we find HCF of $x+1$ and x^2-1

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2-1 \\ & -x^2 \pm x \\ \hline & -x-1 \\ & \quad x \quad 1 \\ \hline & 0 \quad 0 \end{array}$$

The HCF of all the three polynomials is$(x+1)$.

LCM by Division Method

To find the LCM of two or more algebraic expressions (Polynomials) by division method, the following formula is used

$$\text{LCM}=\frac{\text{Product of two polynomials}}{\text{HCF of two polynomials}}$$

Example Find the LCM of $x^3-6x^2+11x-6$ and x^3-4x+3.

Solution: $x^3-6x^2+11x-6$ and x^3-4x+3

Now, find HCF by actual division,

$$\begin{array}{r|l} & \qquad 1 \\ \hline x^3-4x+3 & x^3-6x^2+11x-6 \\ & -x^3 \qquad \mp 4x \pm 3 \\ \hline & \quad -6x^2+15x-9 \end{array}$$

$-6x^2+15x-9=-3(2x^2-5x+3)$, we omit -3.

Now multiply x^3-4x+3 by 2

$$\begin{array}{r|l} & \qquad x+5 \\ \hline 2x^2-5x+3 & 2x^3-8x+6 \\ & -2x^3 \pm 3x \mp 5x^2 \\ \hline & \quad 5x^2-11x+6 \end{array}$$

Multiplying by 2, i.e., $10x^2-22x+12$,

$$\begin{array}{r|l} & 5 \\ 2x^2-5x+3 & 10x^2-22x+12 \\ & -10x^2 \quad 25x \pm 15 \\ \hline & 3x-3 \end{array}$$

$3x-3=3(x-1)$ omit 3 and then again by division

$$\begin{array}{r|l} & 2x-3 \\ x-1 & 2x^2-5x+3 \\ & -2x^2 \quad 2x \\ \hline & -3x+3 \\ & 3x \pm 3 \\ \hline & 0 \quad 0 \end{array}$$

$\therefore$ HCF = x–1.

We know that LCM = $\dfrac{p(x).q(x)}{\text{HCF}}$

$\therefore$ LCM = $\dfrac{(x^3-6x^2+11x-6)(x^3-4x+3)}{x-1}$.

Now divide $x^3-6x^2+11x-6$ by $x-1$, we have,

$$\begin{array}{r|l} & x^2-5x+6 \\ x-1 & x^3-6x^2+11x-6 \\ & -x^3 \quad x^2 \\ \hline & -5x^2+11x \\ & -5x^2 \quad \pm 5x \\ \hline & 6x-6 \\ & -6x \pm 6 \\ \hline & 0 \quad 0 \end{array}$$

Therefore: **LCM** $=(x^3-4x+3)(x^2-5x+6)$.

5.1.4 Solve real life problems related to HCF and LCM

Example 01 Rida has two pieces of cloth one piece is 45 inches wide and other piece is 90 inches wide. She wants to cut both the strips of equal width. How wide should she cut the strips?

Solution: This problem can be solved using HCF because she is cutting or dividing the cloth for widest possible strips.

So,

HCF of 45 and 90

$$45 = 3\times 3\times 5$$

$$90 = 2\times 3\times 3\times 5$$

$$\text{HCF} = \text{Product of common factors}$$

$$\text{HCF} = 3\times 3\times 5$$

$$\text{HCF} = 45$$

So, Rida should cut each piece to be 45 inches wide.

Example 02 Sarfraz exercises every 8 days and Imran every 4 days. Sarfraz and Imran both exercised today. After how many days they will exercise together again?

Solution: This problem can be solved using least common multiple because we are trying to find out the time they will exercise, time that it will occurs out the same time.

LCM of 8 and 4 is

$$8 = 2\times 2\times 2$$

$$4 = 2\times 2$$

$$\text{LCM} = \text{Product of common factors}\times\text{Product of non common factors}$$

$$\text{LCM} = 2\times 2\times 2$$

$$\text{LCM} = 8$$

So, They will exercise together again after 8 days.

Exercise 5.1

1. **Find the HCF of the following expressions by factorization method:**

(i) $72x^4y^5z^2$ and $120x^3y^6z^8$ (ii) $18r^3s^4t^5$, $120r^4s^3t^8$ and $210r^7s^7t^3$

(iii) $x^2-3x-18$ and x^2+5x+6 (iv) $4x^2-9$ and $2x^2-5x+3$

(v) $(2a^2-8b^2)$, $(4a^2+4ab-24b^2)$ and $(2a^2-12ab+16b^2)$

(vi) x^3+x^2+x+1 and x^3+3x^2+3x+1

2. **Find the HCF of the following expressions by division method:**

(i) x^2+3x+2 and $3x^2-3x-6$

(ii) $2x^3+15x^2+31x+12$ and $6x^3+46x^2+100x+48$

(iii) $x^3-5x^2+10x-8$, x^3-4x^2+7x-6

(iv) $x^4+3x^3+2x^2+3x+1$, x^3+4x^2+4x+1 and x^3+5x^2+7x+2

3. **Find the LCM of the following expressions by factorization method:**

(i) $27a^4b^5c^2$ and $81ab^2c^8$ (ii) $24p^2q^3r^4$, $100p^5q^4r^5$ and $300p^3qr^8$

(iii) $21x^2-14x$ and $3x^2-5x+2$ (iv) $x^2+11x+28$ and x^2+x-12

(v) $6x^2+11x+3$, $3x^2-2x-1$ and $3x^2-2x-1$

(vi) x^2-y^2, x^3-y^3 and $x^4+x^2y^2+y^4$

4. **Find the LCM by Division method:**

(i) $x^2-25x+100$ and x^2-x-20

(ii) $3x^2+14x+8$ and $6x^2+x-2$

(iii) $x^2-y^2-z^2-2yz$ and $y^2-z^2-x^2-2xz$

(iv) $3x^3+9x^2-84x$ and $4x^4-24x^3+32x^2$

5. If the HCF of the $x^2-11x+24$ and x^2-6x+6 is $(x-3)$. Find the **LCM.**

6. The **HCF** and **LCM** of two expressions are $(x+3)$ and $(x^3+7x^2+7x-15)$, respectively. If one expression is $x^2+8x+15$. Find the second expression.

7. The HCF and LCM of two polynomial of the second degree are $3x-2$ and $3x^3+7x^2-4$ respectively. Find the product of two polynomial.

8. Verify the relationship between HCF and LCM.
i.e.(HCF×LCM $=$ $p(x)q(x)$) for the polynomial $p(x)=x^2-8x-20$ and $q(x)=x^2-15x+50$

9. A carpenter got some free wooden planks. Some are 12cm long and some are 18cm. He wants to cut them so that he has equal size planks to make using them easier. What size planks should he cut them into to avoid wasting any wood?
10. Train A and train B stops at Hyderabad as 10:30am. Train A stops every 12 minutes and train B stops every 14 minutes. when do they both stop together?

5.2 Basic operations on Algebraic Fractions

If $p(x)$ and $q(x)$ are algebraic expressions and $q(x) \neq 0$ then $\frac{p(x)}{q(x)}$ is called an Algebraic Fraction.

Simplest form of algebraic fraction is a fraction in which there is no common factor except 1 in numerator and denominator. In algebraic fraction fundamental operations $(+,-,\div,\times)$ are carried out in the same way as in common fractions.

In the following examples we shall explain the use of highest common factor and least common multiple to reduce fractional expressions in simplest form involving fundamental operations.

5.2.1 Use highest common factor and least common multiple to reduce fractional expressions involving addition, subtraction, multiplication and division.

Example 01 Simplify: $\frac{x^2-x-6}{2x^2-5x-3}+\frac{1}{4x^2-1}$

Solution:

$$\frac{x^2-x-6}{2x^2-5x-3}+\frac{1}{4x^2-1} = \frac{x^2-3x+2x-6}{2x^2-6x+x-3}+\frac{1}{(2x-1)(2x+1)}$$

$$= \frac{x(x-3)+2(x-3)}{2x(x-3)+1(x-3)}+\frac{1}{(2x-1)(2x+1)}$$

$$= \frac{(x-3)(x+2)}{(x-3)(2x+1)}+\frac{1}{(2x-1)(2x+1)} \quad \text{where } x \neq 3$$

$$= \frac{(x+2)}{(2x+1)} + \frac{1}{(2x-1)(2x+1)}$$

$$= \frac{(x+2)(2x-1)+1}{(2x-1)(2x+1)}$$

$$= \frac{2x^2 - x + 4x - 2 + 1}{(2x-1)(2x+1)} = \frac{2x^2 + 3x - 1}{4x^2 - 1}$$

Example 02 Simplify: $\dfrac{2}{x+2} - \dfrac{x-4}{2x^2 + x - 6}$

Solution:

$$\frac{2}{x+2} - \frac{x-4}{2x^2 + x - 6}$$

$$= \frac{2}{x+2} - \frac{x-4}{2x^2 + 4x - 3x - 6}$$

$$= \frac{2}{x+2} - \frac{x-4}{2x(x+2) - 3(x+2)}$$

$$= \frac{2}{x+2} - \frac{x-4}{(x+2)(2x-3)}$$

$$= \frac{2(2x-3)-(x-4)}{(x+2)(2x-3)}$$

$$= \frac{4x - 6 - x + 4}{(x+2)(2x-3)}$$

$$= \frac{3x-2}{2x^2 + x - 6}$$

Example 03 Simplify: $\dfrac{ab^2 + 2a}{ab - 6 + 2b - 3a} \times \dfrac{b^2 - 6b + 9}{b^3 + 2b}$

Solution:

$$\frac{ab^2 + 2a}{ab - 6 + 2b - 3a} \times \frac{b^2 - 6b + 9}{b^3 + 2b}$$

$$\frac{ab^2 + 2a}{ab - 6 + 2b - 3a} \times \frac{b^2 - 6b + 9}{b^3 + 2b} = \frac{a(b^2+2)}{b(a+2) - 3(a+2)} \times \frac{b^2 - 3b - 3b + 9}{b(b^2+2)}$$

$$= \frac{a}{(a+2)(b-3)} \times \frac{b(b-3) - 3(b-3)}{b}$$

$$= \frac{a}{(a+2)(b-3)} \times \frac{(b-3)(b-3)}{b}$$

$$= \frac{a}{(a+2)(b-3)} \times \frac{(b-3)(b-3)}{b}$$

$$= \frac{a(b-3)}{b(a+2)} \qquad b-3 \neq 0$$

Example 04 Simplify: $\frac{p^2-q^2}{r^2+2rs+s^2} \div \frac{2(p+q)}{3(r+s)s}$

Solution: Simplification $\frac{p^2-q^2}{r^2+2rs+s^2} \div \frac{2(p+q)}{3(r+s)s}$

$$= \frac{(p+q)(p-q)}{(r+s)^2} \div \frac{2(p+q)}{3(r+s)s}$$

$$= \frac{(p+q)(p-q)}{(r+s)^2} \times \frac{3(r+s)s}{2(p+q)} \qquad \text{provided } p+q \neq 0 \text{ and } r+s \neq 0$$

$$= \frac{3s(p-q)}{2(r+s)}$$

Exercise 5.2

Simplify the following

(i) $\frac{4x}{x^2+2x+1} + \frac{3}{x+1}$

(ii) $\frac{3}{x(2x+1)} + \frac{6x+7}{3x(x+1)}$

(iii) $\frac{3x-1}{x^2+2x+1} - \frac{4x^2-1}{x^2-2x-3}$

(iv) $\frac{1}{x+1} - \frac{2}{x+2} + \frac{3}{x+3}$

(v) $\frac{x^2+4x+3}{5} \times \frac{10}{x+1}$

(vi) $\frac{x^2-4x-21}{x^2-6x-7} \div \frac{x+9}{x+1}$

(vii) $\left[\frac{3}{x+1} + \frac{1}{x+2}\right] \div \left[\frac{2}{x+3} - 1\right]$

(viii) $\left(\frac{1}{x^2-9}\right) \div \left(\frac{1}{x+3}\right) - \frac{3}{x-2}$

(ix) $\frac{1}{x^2+8x+15} + \frac{1}{x^2+7x+12} - \frac{1}{x^2+x-12}$

(x) $2\left(\frac{x^2+7x+12}{x^2-16} + \frac{x^2+x-2}{x^2+4x+4}\right) \times \frac{x^2-2x-8}{8x^2+2x+4}$.

5.3 Square Root of an Algebraic Expressions

5.3.1 Find square root of an algebraic expression by Factorization and Division

We shall discuss two methods to find the square roots of the algebraic expressions.

(a) By Factorization Method (b) By Division Method

(a) Square Root by Factorization method.

Example 01 Find square root of the expression $49x^2+126xy+81y^2$ by factorization method.

Solution: $49x^2+126xy+81y^2$

$= (7x)^2 + 2(7x)(9y) + (9y)^2$

$= (7x + 9y)^2$

Therefore $\sqrt{49x^2+126xy+81y^2} = \sqrt{(7x+9y)^2}$

$= 7x + 9y$

(b) Square Root by Division Method.

Example 01 Find square root of the expression $4x^4 + 12x^3 - 19x^2 - 42x + 49$ by division method.

Solution: Method is illustrated below,

$2x^2 + 3x - 7$

$2x^2$	$4x^4 + 12x^3 - 19x^2 - 42x + 49$
$+2x^2$	$\pm 4x^4$
$4x^2 + 3x$	$12x^3 - 19x^2 - 42x + 49$
$+3x$	$\pm 12x^3 \pm 9x^2$
$4x^2 + 6x - 7$	$-28x^2 - 42x + 49$
-7	$\mp 28x^2 \mp 42x \pm 49$
$4x^2 + 6x - 14$	0 0 0

Therefore: $\sqrt{4x^4 + 12x^3 - 19x^2 - 42x + 49} = 2x^2 + 3x - 7.$

Example 02 For what value of a the expression $36x^4 + 36x^3 + 57x^2 + 24x + a$ will be the perfect square?

Solution: By division method, we have,

	$6x^2 + 3x + 4$
$6x^2$	$36x^4 + 36x^3 + 57x^2 + 24x + a$
$+6x^2$	$\pm 36x^4$
$12x^2 + 3x$	$36x^3 + 57x^2 + 24x + a$
$+3x$	$\pm 36x^3 \pm 9x^2$
$12x^2 + 6x + 4$	$48x^2 + 24x + a$
$+4$	$\pm 48x^2 \pm 24x \pm 16$
$12x^2 + 6x + 8$	$a - 16$

Given expression will be a perfect square if

$a - 16 = 0 \Rightarrow a = 16,$

Therefore, $a = 16$, will make the given expression a perfect square.

Example 03 What should be added to the expression $x^4 + 4x^3 + 10x^2 + 5$, so that it may be a perfect square

Solution: By division method, we have,

	$x^2 + 2x + 3$
x^2	$x^4 + 4x^3 + 10x^2 + 5$
x^2	$\pm x^4$
$2x^2 + 2x$	$4x^3 + 10x^2 + 5$
$+2x$	$\pm 4x^3 \pm 4x^2$
$2x^2 + 4x + 3$	$6x^2 + 5$
$+3$	$\pm 6x^2 \pm 12x$
$2x^2 + 4x + 6$	$-12x - 4$ or $-(12x + 4)$

The given expression would be perfect square if remainder vanishes, which is only possible when $(12x + 4)$ is added.

Exercise 5.3

1. **Find the square root of the following algebraic expressions by factorization method.**

(i) $36x^2-60xy+25y^2$

(ii) $9x^2+\frac{1}{x^2}+6$

(iii) $4x^4y^4-\frac{12x^3y^3}{z^2}+\frac{9x^2y^2}{z^4}$

(iv) $36(3-2x)^2-48(3-2x)y+16y^2$

(v) $\left(x^2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)+3$

(vi) $(4x^2-4x+1)(9x^2-54x+81)$

(vii) $(x^2-2x+1)(x^2-6x+9)$

(viii) $(x^2+8x+15)(x^2+7x+10)(x^2+5x+6)$

2. **Find the square root of the following algebraic expressions by division method.**

(i) $x^4+2x^3+3x^2+2x+1$

(ii) $25x^4+40x^3+26x^2+8x+1$

(iii) $4x^4+8x^3+20x^2+16x+16$

(iv) $\frac{x^2}{y^2}+\frac{y^2}{x^2}+47-\frac{14y}{x}+\frac{14x}{y}$

(v) $x^2-2x+3-\frac{2}{x}+\frac{1}{x^2}$

(vi) $x^2+\frac{y^2}{9}+9z^2+\frac{2xy}{3}+2yz+6xz$

(vii) $(x^2+\frac{1}{x^2})^2-8(x^2+\frac{1}{x^2})+16$

(viii) $x^6+\frac{1}{x^6}-4\left(x^3+\frac{1}{x^3}\right)+6,\ x\neq 0$

3. What should be added to $4x^4+4x^3+17x^2+8x+9$ to make it perfect square?

4. What should be subtracted from $9x^6-12x^5+4x^4-18x^3-12x^2+18$ to make it a perfect square?

5. For what value of 'm', $9x^4+12x^3+34x^2+mx+25$ will be the perfect square?

6. For what value of 'p' and 'q', the expression $x^4+8x^3+30x^2+px+q$ will be the perfect square?

7. For what values of 'a' and 'b', $x^4+4x^3+10x^2+ax+b$ will be the perfect square?

Review Exercise 5

1. True and false questions

Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) HCF of $y^2 - 4$ and $y - 2$ is $y - 2$. T/F

(ii) HCF of $a^3 - 1$ and $a^2 - 1$ is $a + 1$. T/F

(iii) LCM of $x^3 + 1$ and $x + 1$ is $x^3 + 1$. T/F

(iv) LCM of $x^4 - y^4$ and $x^2 - y^2$ is $x^2 + y^2$. T/F

(v) HCF of $a^2 + 4a + 3$ and $a^2 + 5a + 6$ is $a + 3$. T/F

2. Fill in the blanks.

(i) There are ______ methods for finding the HCF of polynomials.

(ii) LCM × HCF of two polynomials $p(x)$ and $q(x)$ = ___________.

(iii) HCF of $y^2 - 5y$, $y + 6$ and $y - 2$ is ________.

(iv) LCM of y^2+3y+2 and y^2+5y+6 is ________.

(v) HCF of $y^2 - \frac{1}{y^2}$ and $y + \frac{1}{y}$ is ________.

3. Tick (✓) the correct answers

(i) HCF of $x^3 - 8y^3$ and $x^2 - 4xy + 4y^2$ is:

(a) $x - 4y$ (b) $x^2+2xy+y^2$

(c) $x+2y$ (d) $x-2y$

(ii) LCM of $(2y+3z)^5$ and $(2y+3z)^3$ is:

(a) $(2y+3z)^8$ (b) $(2y+3z)^3$

(c) $(2y+3z)^2$ (d) $(2y+3z)^5$

(iii) HCF of $x^3 - y^3$ and x^2+xy+y^2 is:

(a) $x+y$ (b) x^2+xy+y^2

(c) $x-y$ (d) $(x-y)^2$

(iv) LCM of $(x-y)^4$ and $(x-y)^3$ is:

(a) $(x-y)$ (b) $(x-y)^3$

(c) $(x-y)^4$ (d) $(x-y)^7$

(v) Simplified form of $\frac{1}{x+y}+\frac{y}{x^2-y^2}$ is:

(a) $\frac{y+1}{x^2-y^2}$ (b) $\frac{x}{x^2-y^2}$

(c) $\frac{y}{x^2-y^2}$ (d) $\frac{x+y}{x^2-y^2}$

(vi) Simplified form of $\frac{y}{25x^2-y^2}-\frac{1}{5x-y}$ is:

(a) $\frac{5x}{25x^2-y^2}$ (b) $\frac{5x}{5x-y}$

(c) $\frac{-5x}{5x+y}$ (d) $\frac{-5x}{25x-y^2}$

(vii) $\frac{a^3x^3+a^3y^3}{a^2(x+y)}$ =____________:

(a) ax^2+ay^2 (b) x^2+y^2

(c) $a(x^2-xy+y^2)$ (d) $a(x^2+xy+y^2)$

(viii) $\frac{a}{a-b}-\frac{b}{a+b}$ =____________:

(a) $\frac{a^2+b^2}{a-b}$ (b) $\frac{a^2+b^2}{a^2-b^2}$

(c) $\frac{a+b}{a^2-b^2}$ (d) $\frac{a-b}{a^2-b^2}$

(ix) LCM =________________, given that p and q are any two polynomials.

(a) $\frac{\text{HCF}}{p\times q}$ (b) $\frac{p\times q}{\text{HCF}}$

(c) $\frac{p}{\text{HCF}}$ (d) $\frac{q}{\text{HCF}}$

(x) LCM of x^2-x+1 and x^3+1 is:

(a) $x+1$ (b) x^2-x+1 (c) x^3+1 (d) x^2+x+1

Summary

- There are two methods to be used to find the HCF and LCM of algebraic expression
  (i) Factorization method (ii) Division method
- LCM × HCF of two polynomials = product of polynomials
- With help of LCM and HCF, addition, multiplication, subtraction and division of algebraic expression can be found.
- There are two methods to be used to find the square root of algebraic expression
  (i) Square root by Factorization method
  (ii) Square root Division method

Unit 6

• Weightage = 6%

LINEAR EQUATION AND INEQUALITIES

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Recall linear equations in one variable.
- Solve linear equations with rational coefficients.
- Reduce equations involving radicals, to simple linear form and find their solutions.
- Define absolute values.
- Solve the equations, involving absolute values in one variable.
- Define inequalities (>, <) and (≥, ≤).
- Recognize properties of inequalities (i.e., trichotomy, transitive, additive and multiplicative).
- Solve linear inequalities with rational coefficients.

6.1 Linear Equations

6.1.1 Recall Linear Equation in one Variable:

If symbol of equality "=" is involved in an open sentence then such sentence is called an **equation**. Linear equations with one variable i.e. $ax+b=0$, $a \neq 0$, are equations where variable has an exponent "1" which is typically not shown.

6.1.2 Solve linear equations with Rational Coefficients:

The value of the unknown (variable) for which the given equation becomes true is called a solution or root of the equation.

Example 01 Solve: $3x-1=5$

Solution: $3x-1=5$

$\Rightarrow \quad 3x = 5+1$

$\Rightarrow \quad x = \frac{6}{3}$

$\Rightarrow \quad x = 2$

Thus, the solution set is $\{2\}$

Example 02 Solve: $\frac{2}{3}(x+3) = 3 + \frac{5x}{9}$

Solution: $\frac{2}{3}(x+3) = 3 + \frac{5x}{9}$

$9 \times \frac{2}{3}(x+3) = 9 \times 3 + 9 \times \frac{5x}{9}$ (Multiplying both sides by 9)

$\Rightarrow \quad 3 \times 2(x+3) = 27 + 5x$

$\Rightarrow \quad 6(x+3) = 27 + 5x$

$\Rightarrow \quad 6x + 18 = 27 + 5x$

$\Rightarrow \quad 6x - 5x = 27 - 18$

$\Rightarrow \quad x = 9$

Thus, the solution set is $\{9\}$.

Example 03 Age of father is 13 times the age of his son. It will be only five times after four years. Find their present ages.

Solution: Let present age of son = x years,

and present age of father = $13x$ years,

According to given condition,

$\therefore \quad 13x + 4 = 5(x + 4)$

$\Rightarrow \quad 13x + 4 = 5x + 20$

$\Rightarrow \quad 13x - 5x + 4 = 5x - 5x + 20$ (Subtracting $5x$ from both sides)

$\Rightarrow \quad 8x + 4 = 20$

$\Rightarrow \quad 8x + 4 - 4 = 20 - 4$ (Subtracting 4 from both sides)

$\Rightarrow \quad 8x = 16$

$\Rightarrow \quad \frac{8x}{8} = \frac{16}{8}$ (Dividing 8 on both sides)

$\Rightarrow \quad x = 2$

Hence present age of father $= 13 \times 2 = 26$ years
and present age of son $= 2$ years.

Example 04 When 16 is added to $\frac{1}{3}$ of number the result is $2\frac{1}{3}$ of the original number. Find the number?

Solution: Let x be the number, the according to the given condition:

$16 + \frac{1}{3}x = 2\frac{1}{3}x$

$\Rightarrow \quad 16 + \frac{1}{3}x = \frac{7}{3}x$

$\Rightarrow \quad 16 = \frac{7}{3}x - \frac{1}{3}x$

$\Rightarrow \quad 16 = \left(\frac{7}{3} - \frac{1}{3}\right)x$

$\Rightarrow \quad 16 = \left(\frac{7-1}{3}\right)x$

$\Rightarrow \quad 16 = \frac{6}{3}x$

$\Rightarrow \quad 16 \times 3 = 6x$

$\Rightarrow \quad \frac{48}{6} = x \qquad \Rightarrow x = 8$

6.1.3 Reduce Equations involving radicals to Simple linear Form and find their solutions.

Definition: An equation in which the variable appears under the radical sign, is called the radical equation.

For example, $3\sqrt{t} - \sqrt{t+1} = 2$ and $\sqrt{x} = 8$ are radical equations.

Solution of radical equation is explained with help of the following example.

Example 01 Solve: $\sqrt{2x+11}=\sqrt{3\,x+7}$

Solution: $\sqrt{2x+11}=\sqrt{3x+7}$

Squaring on both the sides, we have,

$$\left(\sqrt{2x+11}\right)^2=\left(\sqrt{3x+7}\right)^2$$

$\Rightarrow$ $2x+11=3x+7$

$\Rightarrow$ $2x-3x=7-11$

$\Rightarrow$ $-x=-4$

$\Rightarrow$ $x=4$

Verification: Put $x=4$ in the given equation, then,

$\sqrt{2(4)+11}=\sqrt{3(4)+7}$

or $\sqrt{8+11}=\sqrt{12+7}$

or $\sqrt{19}=\sqrt{19}$

Thus, solution set is $\{4\}$.

Notes: 1. Sometimes the obtained root from radical equation does not satisfy the original equation, it is called an extraneous root.

2. Solutions of radical equations must be verified.

Exercise 6.1

1. Solve the following equations

(i) $\frac{1}{4}x=5$ (ii) $\frac{x}{4}=-3$ (iii) $-5=\frac{-x}{6}$

(iv) $\frac{-x}{8}=-5$ (v) $y-\frac{2}{5}=-\frac{1}{3}$ (vi) $2y-\frac{3}{5}=\frac{1}{2}$

(vii) $\frac{2x-4}{5}=\frac{5x-12}{4}$ (viii) $\frac{3x}{5}+7=\frac{2x}{3}$ (ix) $\frac{3x}{5}+7=\frac{2x}{3}+\frac{4x}{5}$

(x) $\frac{6}{2x-5}-\frac{4}{x-3}=0$ (xi) $\frac{7x-4}{15}=\frac{7x+4}{10}$

(xii) $\frac{3x-2}{10}=\frac{7x-3}{15}-2$ (xiii) $\frac{12x-3}{12}=\frac{12x+3}{8}$

(xiv) $\frac{1}{4}x+x=-3+\frac{1}{2}x$ (xv) $\frac{1}{3}+2m=m-\frac{3}{2}$

2. When 25 added to a number, the result is halved; the answer is 3 times the original number. What is the number?
3. When a number is added to 4, the result is equal to subtracting 10 from 3 times of it. What is the number?
4. Bilal is 6 year older than Ali, Five years from now the sum of their age will be 40. How old are both of them.
5. **Find the Solution set of the following equations and also verify the answer:**

(i) $6+\sqrt{x}=7$ (ii) $\sqrt{x-9}=1$ (iii) $\sqrt{\frac{y}{4}}-2=3$

(iv) $\sqrt{4x+5}=\sqrt{3x-7}$ (v) $\frac{\sqrt{3y+12}}{7}=3$ (vi) $\sqrt{x}+9=7$

(vii) $\sqrt{25y-50}=\sqrt{y-2}$ (viii) $\sqrt{x}-8=1$ (ix) $10\sqrt{x+20}=100$

6.2 Equations involving absolute values

6.2.1 Define absolute values

The absolute value of a real number x is denoted by $|x|$, is the distance of x from zero, either from left or from right of zero.

If x is real number, then absolute value or modulus value of x is denoted by $|x|$, is defined as under:

$$|x| = \begin{cases} x, \text{ when } x > 0 \\ 0, \text{ when } x = 0 \\ -x, \text{ when } x < 0 \end{cases}$$

Example $|-5|=5, |+7|=7, \left|-\frac{1}{2}\right|=\frac{1}{2}, |0|=0$ and so on.

Note: The absolute value of a number is always non negative.

6.2.2. Solve the Equations, involving absolute values in one variable

Example 01 Find the solution set of $|5x-3|-2=3$

Solution: Given that

$$|5x-3|-2=3$$

$$\Rightarrow \quad |5x-3|=5$$

By the definition of modulus, we have,

$5x-3=5$ or $5x-3=-5$

$\Rightarrow$ $5x=5+3$ or $5x=-5+3$

$\Rightarrow$ $5x=8$ or $5x=-2$

$\Rightarrow$ $x=\dfrac{8}{5}$ or $x=-\dfrac{2}{5}$

Thus, the solution set is $\left\{\dfrac{8}{5}\ ,\ -\dfrac{2}{5}\right\}$

Example 02 Find the solution set of $|5x-3|+7=3$

Solution: Given that

$$|5x-3|+7=3$$

$$\Rightarrow |5x-3|=-4$$

The modulus of a real number never be negative

$\therefore$ Solution set = { }

Example 03 Solve $|5x-3|-2=3$, where $x \in W$.

Solution: Given that,

$$|5x-3|-2=3$$

$$\Rightarrow |5x-3|=5$$

By the definition of modulus, we have,

so, $5x-3=5$ or $5x-3=-5$

$\Rightarrow$ $5x=5+3$ or $5x=-5+3$

$\Rightarrow$ $5x=8$ or $5x=-2$

$\Rightarrow$ $x=\dfrac{8}{5}$ or $x=-\dfrac{2}{5}$

$-\dfrac{2}{5}$ and $\dfrac{8}{5} \notin W$

Thus, the solution set is { }

Example 04 Solve $|2y-5|+2=7$

Solution: Given that $|2y-5|+2=7$

$\Rightarrow \quad |2y-5| = 7-2$

$\Rightarrow \quad |2y-5| = 5$

By definition of modulus we have,

so, $2y-5=5$ or $2y-5=-5$

$\Rightarrow \quad 2y = 5+5$ or $2y=-5+5$

$\Rightarrow \quad 2y=10$ or $2y=0$

$\Rightarrow \quad y=\frac{10}{2}$ or $y=\frac{0}{2}$

$\Rightarrow \quad y=5$ or $y=0$

Thus, the solution set is {5, 0}.

Exercise 6.2

Find the solution set of the following equations.

1. $|2x+1|=6$
2. $|5x-12|=7$, where $x \in W$
3. $\left|\frac{2x}{7}\right| = 12$
4. $\left|\frac{2x+1}{3}\right| = 8$
5. $|5x-3|-8=4$, where $x \in N$
6. $\left|\frac{5x+1}{7}\right| - 3 = 8$
7. $\left|\frac{2x+3}{4}\right| + 2 = 7$
8. $\left|\frac{3x+6}{12}\right| + 1 = 3$, where $x \in Z$
9. $\frac{3}{2} = |7x+8|$
10. $\left|\frac{2x-3}{5}\right| - 12 = 5$
11. $|3x+1|+1=\frac{3}{4}$
12. $\left|\frac{2x+1}{7}\right| = 1$

6.3 Linear inequalities

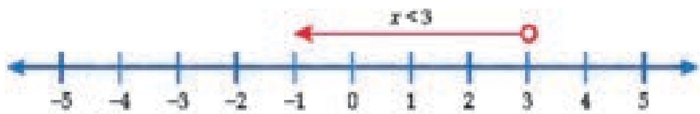
A linear algebraic expression which contains the sign of inequality is called linear inequality or linear inequation.

6.3.1 Define inequalities (>, <) and (≥ , ≤).

The following relational operators are called inequilities.

‘<’ means less than,
‘>’ means greater than,
‘≤’ means less than or equal to,
‘≥’ means greater than or equal to.

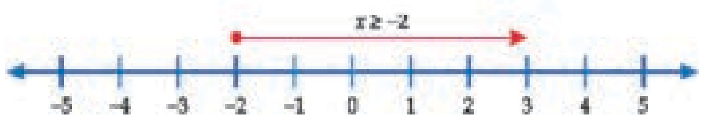
Example 01 Illustrate $x < 3$ on the number line



Example 02 Illustrate $x > -3$ on the number line

Example 03 Illustrate $x \leq 4$ on the number line

Example 04 Illustrate $x \geq -2$ on the number line



Note: Hollow circle '○' shows that number is not included and dark circle '●' shows that number is included.

6.3.2 Recognize properties of inequalities (trichotomy, transitive, additive, multiplicative).

The following are some important properties of inequalities.

(i) Trichotomy Property:

For any two real numbers a and b, one and only one statement of the following is always true.

$a < b, a = b$ or $a > b$

(ii) Transitive Property:

For any three real numbers a ,b and c

If $a < b$ and $b < c \Rightarrow a < c$

and $a > b$ and $b > c \Rightarrow a > c$

(iii) Additive Property:

For any three real numbers

if $a > b$ then $a + c > b + c,\ \ \forall\ a, b, c \in \mathbb{R}$

or if $a < b$ then $a + c < b + c,\ \ \forall\ a, b, c \in \mathbb{R}$

(iv) Multiplicative Property:

(a) If $a > b$ then $ac > bc,\ \ \forall a, b, c \in \mathbb{R}$ and $c > 0$

or If $a < b$ then $ac < bc,\ \ \forall a, b, c \in \mathbb{R}$ and $c > 0$

(b) If $a > b$ then $ac < bc,\ \ \forall a, b, c \in \mathbb{R}$ and $c < 0$

or If $a < b$ then $ac > bc,\ \ \forall a, b, c \in \mathbb{R}$ and $c < 0$

6.4 Solving linear Inequalities

6.4.1 Solving linear inequalities with rational coefficients.

The following examples will help us understand the solution and show on the number line.

Example 01 Find the solution set of $3x + 1 < 7\ \ \forall x \in W$, and show on the number line

Solution: Given that

$$3x + 1 < 7 \qquad \forall x \in W$$

$$(3x + 1) - 1 < 7 - 1$$

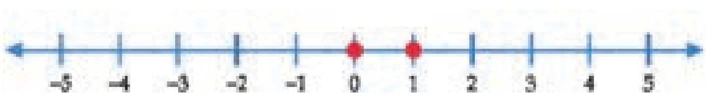
$$3x < 6$$

$$x < \frac{6}{3}$$

$$x < 2$$

Therefore, the solution set is $\{x \mid x \in W \wedge x < 2\} = \{0, 1\}$

The solution is illustrated on the number line as under:

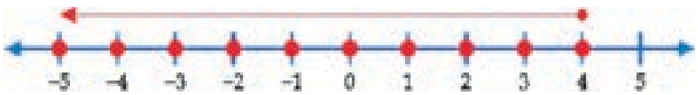


Example 02 Find the solution set of $x-11\le 9-4x\ \forall x\in Z$ and show on the number line

Solution: Given that $x-11\le 9-4x\ \ \forall x\in Z$

$$x-11\le 9-4x$$
$$x+4x\le 9+11$$
$$5x\le 20$$
$$x\le \frac{20}{5}$$
$$x\le 4$$

Therefore, the solution set is $\{x\,|\,x\in Z\wedge x\le 4\}=\{\cdots,-2,-1,0,1,2,3,4\}$



Example 03 Find the solution set of $2x+5>7\ \forall x\in \mathbb{R}$. Also illustrate the solution on the number line.

Solution: Given that

$$2x+5>7\quad x\in \mathbb{R}$$

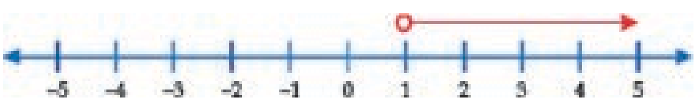
or $2x>7-5$

or $2x>2$

or $x>1$

Thus, the solution set is $\{x\,|\,x\in \mathbb{R}\wedge x>1\}$

The solution on number line is illustrated as under:



Example 03 Find the solution set of $-6<2x+1<11,\ \forall x\in Z$. Also express it on number line.

Solution: Given that $-6<2x+1<11,\ \forall x\in Z.$

Splitting the inequality as under:

$-6<(2x+1)$ and $2x+1<11$

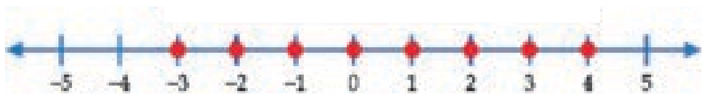
or $-6-1<2x$ and $2x+1<11-1$

or $-7<2x$ and $2x<10$

or $\frac{-7}{2}<x$ and $x<5$

Thus, the solution set is $\{x|x\in Z\wedge -\frac{7}{2}<x<5\}=\{-3,-2,-1,0,1,2,3,4\}$

The solution on number line is illustrated as under:



Example 04 Ayesha scored 78,72 and 86 on the first three out of four tests. What score must be recorded on the fourth test to have average at least of 80?

Solution: let score of the fourth test be x so that.

$$\frac{78+72+86+x}{4}\geq 80$$

$$78+72+86+x\geq 320$$

$$236+x\geq 320$$

$$x\geq 320-236$$

$$x\geq 84$$

Ayesha must score 84 on the fourth test to maintain average of 80.

Exercise 6.3

1. **Find the solution sets of the following inequdities and also illustrate the solution on the number line.**

(i) $2x-7>6+x\quad \forall x\in \mathbb{N}$ (ii) $7x-6>3x+10,\quad \forall x\in \mathbb{R}$

(iii) $\frac{y+5}{20}<\frac{25-4y}{10},\forall y\in \mathbb{N}$ (iv) $|2x+3|<x+2,\forall x\in Z$

(v) $|2y+8|<11,\forall y\in \mathbb{R}$ (vi) $5(2y-3)>6(y-8),\forall y\in \mathbb{R}$

2. Ali scored 66 and 72 marks respectively. For his two Tests, what is the lowest mark he must have scored for his third test If an average score of at least 75 is required to qualify for a bonus prize

3. Seven less than three times the sum of a number and 5 is at least 10, Find all the number that satisfy this condition.

Review Exercise 6

1. True and false questions

Read the following sentences carefully and encircle 'T' in case of true and 'F' in case of false statement.

(i) $ay + b = 0$, where $a = 0$ is a linear equation T/F

(ii) The solution set of $3y - 2 < 7, y \in \mathbb{N}$ is $\{4, 5, 6, ...\}$ T/F

(iii) The solution set of $\sqrt{y} + 1 = 3$ is $\{4\}$. T/F

(iv) The solution set of $|4y| = 8$ is $\{2, -2\}$. T/F

(v) The solution set of $-2 \le x \le 2, x \in Z$ is $\{-2, 0, 2\}$. T/F

2. Fill in the blanks.

(i) The solution set of $2y = -y$ is____________.

(ii) The solution set of $\sqrt{y+5} = 5$ is____________.

(iii) The solution set of $|x| - 4 = 0$ is ____________.

(iv) The solution set of $\sqrt{x+5} + 2 = 4$ is____________.

(v) The solution set of $0 < y + 2 < 5$ when $y \in \mathbb{R}$ is____________.

3. Tick (✓) the correct answer

(i) The solution set of linear equation in one variable has

(a) One solution (b) Two solutions
(c) Three solutions (d) More than one solutions

(ii) $|-20|$

(a) $= 20$ (b) < 20
(c) $= -20$ (d) > 20

(iii) $x \le 4$ means

(a) $x < 4$ (b) $x = 4$
(c) $x < 4$ or $x = 4$ (d) $x > 4$ or $x = 4$

(iv) The solution set of $\sqrt{y} = 10$ is

(a) {100} (b) {10}
(c) {–10} (d) {–10,10}

(v) $\sqrt{y+4} + 2 = 8$ is a

(a) Linear equation (b) Radical equation
(c) Cubic equation (d) Quadratic equation

(vi) The solution set of $5 - 3y = -7$ is

(a) $\{-4\}$ (b) $\{1, 4\}$

(c) $\{4\}$ (d) $\{12\}$

(vii) The solution set of $\sqrt{5y+5} + 5 = 10$ is

(a) $\{\pm 4\}$ (b) $\{5\}$

(c) $\{4\}$ (d) $\{-4\}$

(viii) The solution set of $\left|\frac{5y}{3}\right| = 5$ is

(a) $\{3\}$ (b) $\{-5, 5\}$

(c) $\{3, -3\}$ (d) $\{-3\}$

(ix) The solution set of $|-y| = 0$ is

(a) $\{1\}$ (b) $\{-1\}$

(c) $\{0\}$ (d) $\{\ \}$

(x) If $x > 0$, $y > 0$ and $x - y < 0$, then which of the following relation holds good?

(a) $x < y$ (b) $x + y < 0$

(c) $x > y$ (d) $y - x < 0$

Summary

- An equation of the form $ax + b = 0$, where $a,b \in \mathbb{R}$ and $a \neq 0$ is called a linear equation.
- An equation in which the variable appears under the radical sign, is called a radical equation. Radical equation can have extraneous roots, hence verification of the solution is essential.
- If $x \in \mathbb{R}$ then, $|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x & < 0 \end{cases}$
- If $x,y \in \mathbb{R}$ then
  (i) $|x| \geq 0$ (ii) $|-x| = |x|$ (iii) $|xy| = |x| \cdot |y|$
  (iv) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ (v) $|x| = b$ then $x = b$ or $x = -b$
- For inequality, we use $<, >, \leq, \geq$.
- A linear algebraic expression which contain the sign of inequality is called linear inequality or inequation.
- Properties of inequalities:
  (i) $a < b$ or $a = b$ or $a > b, \quad \forall a,b \in \mathbb{R}$ (Trichotomy)
  (ii) $a > b$ and $b > c \Rightarrow a > c, \quad \forall a,b,c \in \mathbb{R}$ (Transitive)
  (i) $a > b, c > 0 \Rightarrow ac > bc$ and $\frac{a}{c} > \frac{b}{c}, \quad \forall a,b,c \in \mathbb{R}$ (Multiplication and Division Properties)

Unit 7

• Weightage = 5%

LINEAR GRAPH AND THEIR APPLICATION

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Identify pair of real numbers as an ordered pair.
- Recognize an ordered pair through different examples; for instance, an ordered pair (2, 3) to represent a seat, located in an examination hall, at the intersection of 2^{nd} row and 3^{rd} column.
- Describe rectangular/Cartesian plane consisting of two number lines intersecting at right angle at a point 'O'.
- Identify origin O and co-ordinate axes (Horizontal and Vertical axis or x-axis and y-axis respectively) in rectangular plane.
- Locate an ordered pair (a,b) as a geometrical point in the rectangular plane and recognize:
- 'a' as the x- co-ordinate (or abscissa),
- 'b' as the y- co-ordinate (or ordinate).
- Draw different geometrical shapes (e.g., line segment, triangle and rectangle etc.) by joining a set of given points.
- Construct a table for pairs of values satisfying a linear equation in two variables.
- Plot the pairs of points to obtain the graph of a given expression.
- Choose an appropriate scale to draw a graph.
- Draw the graph of:
  - an equation of the form y = c.
  - an equation of the form x = a.
  - an equation of the form y = mx.
  - an equation of the form y = mx + c.
- Draw a graph from the given table of values for x and y.
- Solve applied real life problems.
- Interpret conversion graph as a linear graph relating to two quantities which are in direct proportions.
- Read a given graph to know one quantity corresponding to another.
- Read the graph for conversion of the forms:
  - Miles and kilometers,
  - Acres and hectares,
  - Degrees Celsius and Fahrenheit,
  - Pakistani currency and other currencies, etc.
- Solve simultaneous linear equations in two variables by graphical method.

7.1 Cartesian Plane and Linear Graphs

7.1.1 Identify pair of real number as an ordered pair.

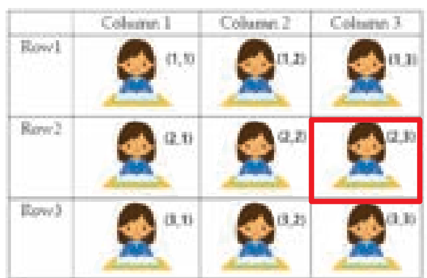
An ordered pair is a pair of two real numbers written in fix order within parenthesis. It helps to locate position of any object in two dimensional space.

7.1.2 Recognize an Ordered Pair Through Different Examples; for instance, an Ordered Pair (2,3) to represent a seat, located in an examination hall, at the intersection of 2nd row and 3rd column:

Let's see the following examples in our surrounding; because of these examples we can recognize the position of an object through rows and columns i.e. form an ordered pair. An ordered pair represents the position of an object or place.

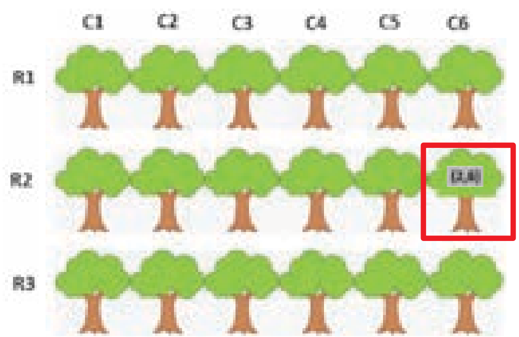
Example 01

The ordered pair (2,3) represents the position of the student in an examination hall as 2nd row and 3rd column. Likewise, every student in hall is located with a unique ordered pair.



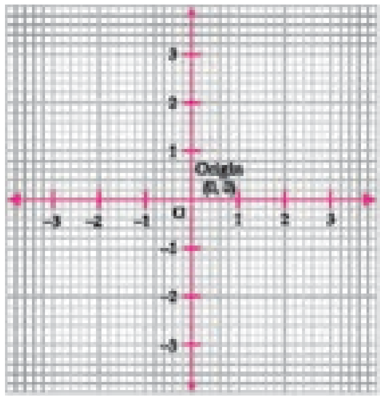
Example 02

If a farmer has planned trees in a garden at equal distances, then (2, 6) represents the tree located in 2nd row and 6th column in the garden.



7.1.3 Describe Rectangular/Cartesian Plane Consisting of Two Number Lines Intersecting at Right Angle at a Point 'O'.

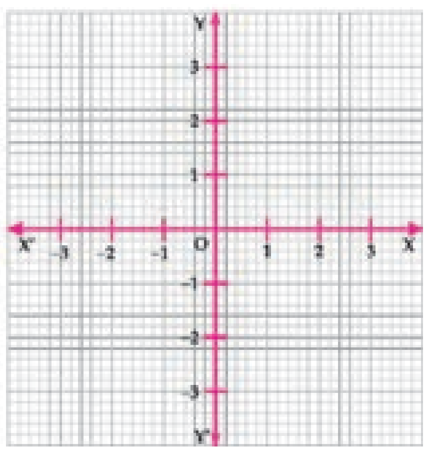
The **Cartesian (rectangular) coordinate system** consists of two real number lines that intersect at a right angle at a point O. These two number lines define a flat surface called a **Cartesian plane**.



7.1.4 Identify Origin 'O' and Coordinate Axis (Horizontal and Vertical axis or *x*-axis or *y*-axis respectively) in Rectangular Plane.

In Cartesian coordinate system, the horizontal number line is called the ***x*-axis** and the vertical number line is called ***y*-axis,** the point where both lines intersect is called origin and it is denoted by **O**.

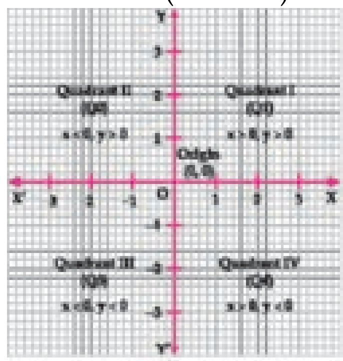
In the given figure the horizontal line $X'X$ is ***x*-axes** and the vertical line $Y'Y$ is the *y*-axes of given Cartesian plane. The point where both line meet is origin of the plane that is 'O'.



7.1.5 Locate an Ordered pair (*a*,*b*) as a Geometrical Point in the Rectangular Plane and recognize:

- **'*a*' as the *x*-coordinate (abscissa)**
- **'*b*' as the *y*-coordinate (ordinate)**

In general, any point in the Cartesian plane can be represented by the ordered pair (a, b), where 'a' is the x- co-ordinate (abscissa) and 'b' is the y-coordinate (ordinate).



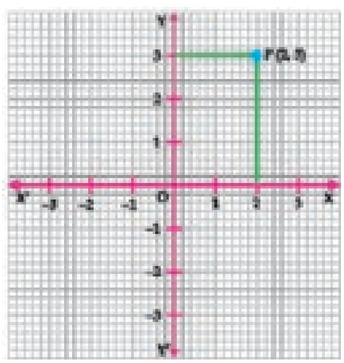
To locate a point in the plane, we must know its x-coordinate, which is its horizontal distance from y-axis and the y-co-ordinate which is its vertical distance from x-axis.

The x-axis and y-axis divide the Cartesian plane into four quadrants named in Roman numbers I, II, III and IV. The Cartesian plane is also known as xy-plane.

- In quadrant I, both x and y-coordinates are positive i.e. $x > 0$ and $y > 0$.
- In quadrant II, x-coordinate is negative and y- co-ordinate (ordinate) is positive i.e. $x < 0$ and $y > 0$.
- In quadrant III, both x and y- coordinate are negative i.e. $x < 0$ and $y < 0$.
- In quadrant IV, x- coordinate is positive and y- coordinate is negative i.e. $x > 0$ and $y < 0$.
- At the origin $x = y = 0$, so the origin has coordinates (0, 0).

Procedure of Graphing a Point in the Cartesian Plane (*xy*-Plane)

Let us learn, how to plot a point in the Cartesian plane through the following example.



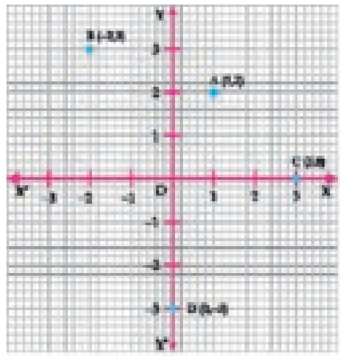
To plot the point (2, 3) in the xy-plane, start from origin and move 2 units to the right of y-axis and then move 3 units up from x-axis as shown in the figure. We reached at the point P which represents (2,3).

Example 01 Determine abscissa and ordinate of the following points. Mention the quadrant in which the point lies also locate the points.

i. A (1, 2) ii. B (-2, 3)
iii. C (3, 0) iv. D (0, -3)

Solution:

Scale: 5 small square = 1 unit



i. A (1, 2)

Here, the abscissa is 1 unit and ordinate is 2 unit, since $x > 0$ and $y > 0$, so, the given point lies in the quadrant I.

ii. B (–2, 3)

Here, the abscissa is -2 and ordinate is 3. Since $x < 0$ and $y > 0$, so, the given point lies in the quadrant II.

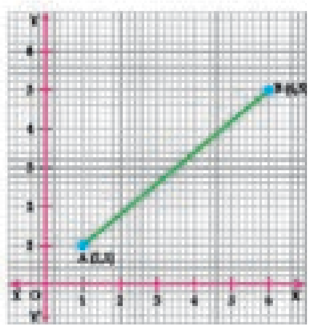
iii. C (3, 0)

Here, the abscissa is 3 unit and ordinate is 0 unit, therefore the point C lies on the x-axis, i.e. right of Origin.

iv. D(0, -3)

Here, the abscissa is 0 unit and ordinate is -3 units, therefore the point D lies on the y-axis, i.e. below the origin.

7.1.6 Draw Different Geometrical Shapes (line segment, triangle and rectangle etc.)



In the previous classes we have studied about the content of several geometrical shapes and also studied how to draw them according to the given information. Now we will draw different geometrical shapes on the graph paper by using the given points.

Example 01 Draw a line segment AB whose ends are A(1,1) and B(6,5).

Method:

Scale: 5 small squares = 1 unit

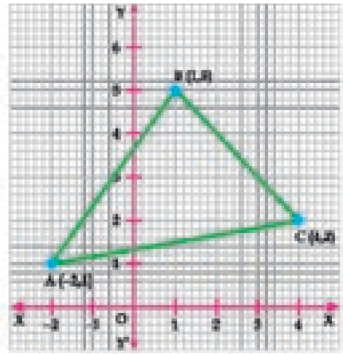
First we locate the points of line on the graph paper then we join them to obtain the line segment AB as shown in the figure.

Example 02 Draw a triangle ABC whose vertices are A (-2,1), B(1,5) and C(4, 2).

Method:

Scale: 5 small squares = 1 unit

First, we plot the points A, B, and C on the graph paper. Then we join them to get the ΔABC as shown in figure.



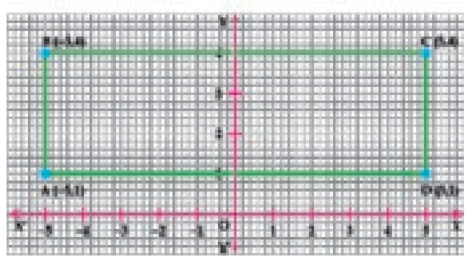
Example 03 Draw a rectangle ABCD whose vertices are A (-5, 1), B (-5, 4), C (5, 4) and D (5, 1).

Method:

Scale: 5 small squares= 1unit.

Plot the points A (-5, 1), B (-5, 4), C(5, 4) and D(5, 1) on the graph paper.

Join the points A to B, B to C, C to D and D to A on the graph paper which results ABCD rectangle.



7.1.7 Construction of table for pairs of values satisfying a linear equation in two variables

This can be explained with the help of below example:

Example 01 Let $x+y=5$ be a linear equation in two variables x and y respectively. Construct the table for some values of x and y.

Solution: Given that $x+y=5$, which can be written as: $y=5-x$

Now prepare the table, put the values of x and get their corresponding values of y.

x	-3	-2	-1	0	1	2	3	…
y	8	7	6	5	4	3	2	…

Exercise 7.1

1. **Determine x and y co-ordinate in the following points:**
   i. A(-2, 2) ii. B(5, -1) iii. C(4, 0)
   iv. D(-5, -6) v. E(3, 4) vi. F($-\sqrt{8}$, $\sqrt{8}$)
2. **Mention the quadrant in which each of the following point lies.**
   i. A(2, -1) ii. B(-3, 3) iii. C($2\sqrt{2}$, $-2\sqrt{2}$)
   iv. D(-2, -4) v. E(5, 4) vi. F$\left(\frac{3}{2}, \frac{5}{2}\right)$
3. **Plot the following points A, B, C and D in the xy-plane.**
   i. A(2, 1), B(3, 2), C(-3, 4), D(-4, -5)
   ii. A(2, 0), B(0, 2), C(3, -3), D(-3, 3)
   iii. A(0, 0), B(-3, -3), C(5, -6), D(-6, 5)
4. Draw a line segment AB by joining the points A(4, 6) and B (-6, 8).
5. Draw a triangle ABC by joining the points A(-1,4), B(-3,-6) and C(3,-2).
6. Draw a rectangle ABCD by joining the points A(0, -1), B(0, 5), C(7, –1) and D(7, 5).
7. Draw a square OABC by joining the points O(0, 0), A(5, 0), B(5, 5) and C(0, 5).
8. Draw a parallelogram OABC whose vertices are O(0, 0), A(2, 4), B(0, 5) and C(6, 4).
9. **Construct the table for some values of x and y of the following linear equations.**
   i. $x+y-2=0$ ii. $2x-y-2=4$
   iii. $\frac{1}{2}(x+2y)-6=0$ iv. $\frac{2}{3}(x-2y)=-2$

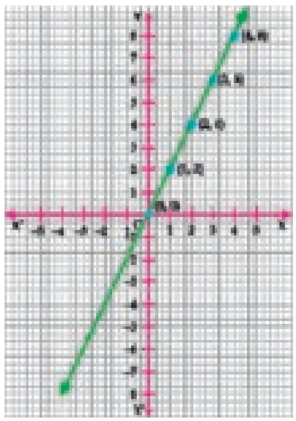
7.1.8 Plot the Pairs of Points to Obtain the Graph of Given Expression

Suppose the linear equation y=2x consist of two variables. Here x is called independent variable and y is called the dependent variable. Because the value of y depends on the value of x.

To create a graph of the given equation we construct a table of values for x and y, and then plot these ordered pairs on the coordinate plane. Two points are enough to determine a line. However, it's always a good idea to plot more than two points to avoid possible errors.

x	0	1	2	3	4	…
y	0	2	4	6	8	…

By continue adding ordered pairs (x,y) in the graph where y-value is the twice of the x-value. Then we draw a line through the points to show all of the points that are on the line. The arrows at each end of the graph indicate that the line continues endlessly in both directions. The resulting graph will look like as shown in the given figure.



7.1.9 Choose an Appropriate Scale to Draw a Graph

Scales should be chosen in such a way that data are easy to plot and easy to read. To determine the numerical value for each grid unit that best fits the range of each variable.

To draw the graph of an equation we choose a scale e.g. 1 small square length represents 1 unit or 2 small squares represent 1 unit etc. It is selected by keeping in mind the size of the paper. Some time the same scale is used for both x and y coordinates and some time we use different scales for x and y coordinate depending on the value of coordinates.

7.1.10 Draw the Graph of:

- An equation of the form $y=c$
- An equation of the form $x=a$
- An equation of the form $y=mx$
- An equation of the form $y=mx+c$

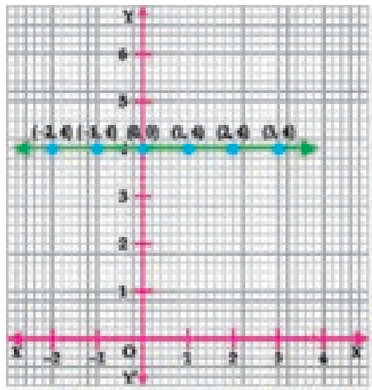
7.1.10 (i) To Draw the Graph of the Equation of the Form $y=c$

Example 01 Draw the graph of $y=4$.

Scale: 5 small squares = 1 unit

Solution: Construct table for drawing graph.

x	-3	-2	-1	0	1	2	3	…
y	4	4	4	4	4	4	4	…



Graph of the equation $y=4$ is shown in the figure.

Note:
Graph of $y = c$ is parallel to x-axis

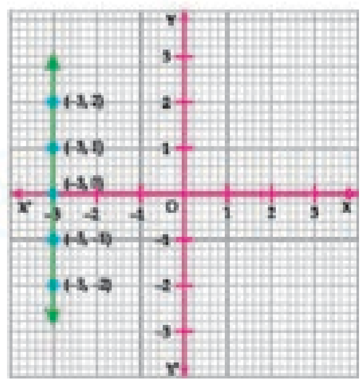
7.1.10(ii) To Draw the Graph of the Equation of the Form $x=a$

Example 01 Draw the graph of the equation $x=-3$

Method: Scale: 5 small squares = 1 unit

Construct table for drawing graph.

x	-3	-3	-3	-3	-3	…
y	-2	-1	0	1	2	…



Graph of the equation is shown in the figure.

Note:
Graph of $x = a$ is parallel to y-axis

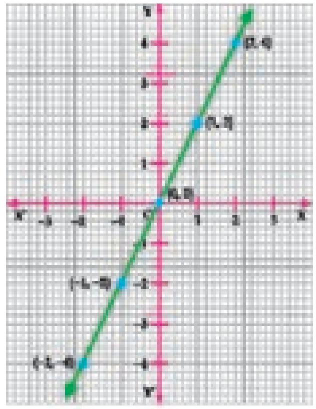
7.1.10(iii) To Draw a Graph of the Equation of the Form $y=mx$

In the equation $y=mx$, the value of y (or its y-coordinate) is the multiple of m and the value of x, where m is constant real number.

Example 01 If $y=2x$, find the value of 'm' and draw its graph.

Solution:

Scale: 5 small square = 1unit
Construct table for drawing graph.



x	-2	-1	0	1	2	...
y	-4	-2	0	2	4	...

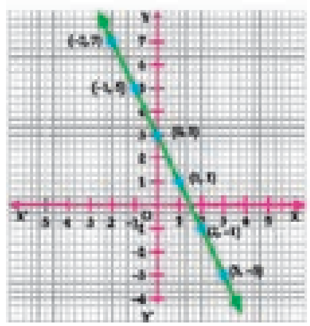
Note:
Graph of $y = mx$ always passes through the origin.

7.1.10(iv) To Draw the Graph of the Equation of the Form $y=mx+c$

In the equation $y=mx+c$, where m and c are any real numbers, and

Example 01 Draw the graph of equation $y = -2x+3$.
Scale: 3 small squares = 1 unit

Method: Construct table for drawing graph.



x	-2	-1	0	1	2	3	...
y	7	5	3	1	-1	-3	...

Note:
Graph of $y = mx+c$ always cut the y-axis at y=c .

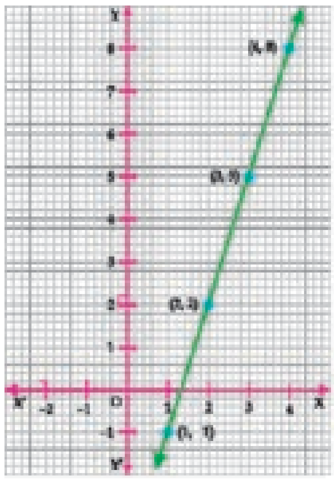
7.1.11 To Draw a Graph from the Given Table of (discrete) values.

In order to draw a graph from a given table of (discrete) values, the values of x and y are combined in the form of the points which are then plotted on the graph paper.

Example 01 Draw the graph of the values of the points given in the table below.

x	1	2	3	4	…
y	-1	2	5	8	…

Method: From the given table, we have, A(1, -1), B(2,2), C(3, 5) and D(4,8), so, we plot the graph on the graph paper.
Scale: 5 small squares = 1 unit



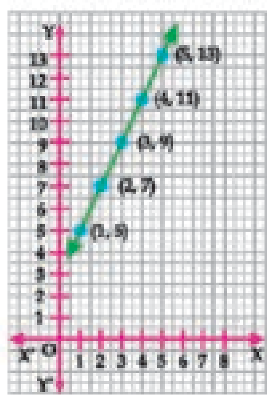
7.1.12 Solve Applied Real Life Problems

Linear equations can be used to model a number of real-life problems, like how much money you make over a time, or the distance that a bicyclist will travel given steady rate of pedaling. Graphing these relationships on a coordinate plane can often help you think about the problem and their solution.

Example

The weight (y) in kg and age (x) in years of a person expressed by the equation $y=2x+3$, draw the age weight graph of the equation.

x(age)	1	2	3	4	5	…
y(weight)	5	7	9	11	13	…



Exercise 7.2

1. Construct a table for the equation given below, satisfying the pair of values: $x + y = 6$

2. Draw the graph from the table given below, using suitable scale.

x	0	-1	4	-4
y	2	4	5	-5

3. Plot the graph of:

i. $y = 3$ ii. $x = 3$ iii. $y = 0$

iv. $y = 2x + 3$ v. $x = 3.5$ vi. $-y = 2x$

4. Find the missing coordinates in the table given below.

S.No	Equation	x-coordinates	y-coordinates
(i)	$y = \frac{1}{2}x$		0
		4	
(ii)	$x = \frac{2}{3}y$	1	
			$\frac{3}{2}$
(iii)	$2x + 4y = 8$	0	
			$\frac{1}{4}$
(iv)	$2x + y = 6$	1	
			0
(v)	$x - y = 2$		0
		1	
(vi)	$x - 3y = 6$	3	
			-1

5. The weight (y) in kg and age (x) in years of a person expressed by the equation $y=2x$. Draw the Age – Weight graph.

6. Ayesha can drive a two-wheeler continuously at the speed of 20km/hour. Construct a distance-time graph for this situation. Through the linear graph calculate:
   a) The time taken by Ayesha to ride 100km.
   b) The total distance covered by Ayesha in 3 hours.

7.2 Conversion Graphs

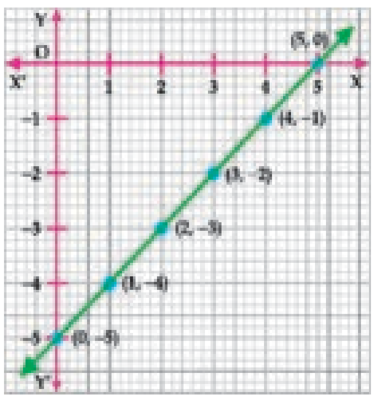
7.2.1 Interpret conversion graph as a linear graph relating to two quantities which are in direct proportions

Here we consider the conversion graph as a linear graph of two quantities which are related in direct proportion.

We demonstrate the ordered pairs which lies on the graph of the equation $y = x - 5$, are calculated values given below table:

x	0	1	2	3	4	5
y	-5	-4	-3	-2	-1	0
(x, y)	(0, -5)	(1,-4)	(2, -3)	(3, -2)	(4, -1)	(5, 0)

Locate the points on the graph for the given linear equation in which for every unit change in x coordinate value there is proportional change in y-coordinate value.



7.2.2 Read a Given Graph to Know One Quantity Corresponding to Another:

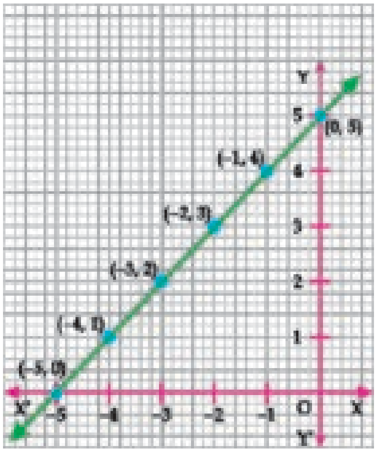
Consider the linear equation $y = x - 5$

For the given values of x we can read the corresponding value of y with the help of: $y = x - 5$

For the given values of y we can read the corresponding value of x, by converting the equation $y = x - 5$ to the equation $x = y + 5$ and draw the corresponding conversion graph. In the conversion graph we express x in term of y as given below: $x = y + 5$

y	-5	-4	-3	-2	-1	0
x	0	1	2	3	4	5
(y, x)	(-5, 0)	(-4, 1)	(-3, 2)	(-2, 3)	(-1, 4)	(0, 5)

The conversion graph of x.r.t. y is drawn on the graph paper.



7.2.3 Read the graph for conversion of the forms:

- Miles and kilometers
- Acres and hectares
- Degrees Celsius and Fahrenheit
- Pakistani currency and other currencies, etc.

If both quantities are in a relation either is increasing or decreasing, then the graph of the relation will be the straight line showing the levels of both quantities indicated by co-ordinate axes.

(i) Read the Graph for Conversion of Miles and Kilometers

Let us discuss the graph of the form of miles and kilometers, both are the units of the distance. If the distance in miles indicated along x-axis and the distance in kilometers along y-axis. Let's see the following examples.

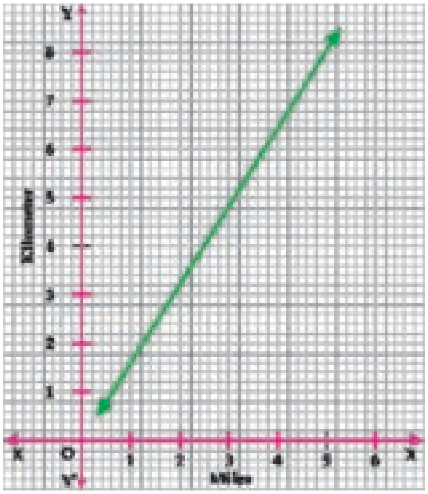
Example Read the following conversion graph between miles and kilometer to approximately convert:

i) 2 miles to kilometers
ii) 5 miles to kilometers
iii) 3 kilometers to miles
iv) 7 kilometers to miles

By using given scales.

Conversion graph between miles and kilometer

Scale: 5 small squares = 1 mile along x-axis
5 small squares = 1 kilometer along y-axis



Solution:

i) 2 miles to kilometers

By reading the above graph as per given scale we can see
2 miles $\cong$ 3.20 kilometers

ii) 5 miles to kilometers

By reading the above graph as per given scale we can see
5miles $\cong$ 8 kilometers

iii) 3 kilometer to miles

By reading the above graph as per given scale we can see
3 kilometers $\cong$ 1.8 miles

iv) 7 kilometer to miles

By reading the above graph as per given scale we can see
7 kilometers $\cong$ 4.20

(ii) Read the Conversion of Hectares into Acres:

Let us discuss the graph of the form of Hectares and Acres, both are the units of land area. If the distance in Hectares indicated along x-axis and the distance in Acres along y-axis. Let's see the following examples.

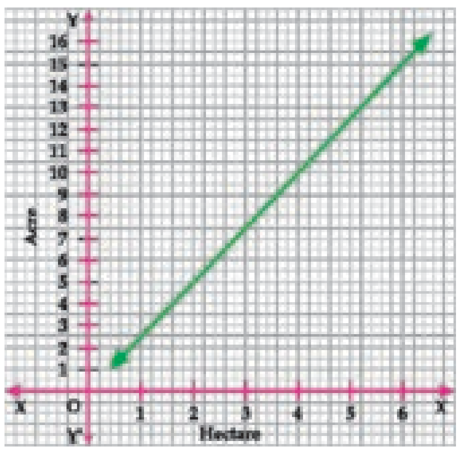
Example: Read the following conversion graph between Hectares and Acres to approximately convert:

i) 2 hectares to acres
ii) 6 hectares to acres
iii) 10 acres to hectares
iv) 8 acres to hectares

By using given scales.

Conversion graph between hectares and acres

Scale: 5 small squares = 1 hectares along x-axis
2 small squares = 1 acres along y-axis



Solution:

i) 2 hectares to acres

By reading the above graph as per given scale we can see
2 hectares $\cong$ 5 acres

ii) 6 hectares to acres

By reading the above graph as per given scale we can see
6 hectares $\cong$ 15 acres

iii) 10 acres to hectares

By reading the above graph as per given scale we can see
10 hectares $\cong$ 4 acres

iv) 8 acres to hectares

By reading the above graph as per given scale we can see

8 kilometers ≅ 3.20

(iii) Read the Conversion graph of Degrees Celsius into Degrees Fahrenheit:

Let us discuss the graph of the form of Celsius and Fahrenheit, both are the units of temperature. If the temperature in Celsius indicated along *x*-axis and the temperature in Fahrenheit along *y*-axis. Let's see the following examples.

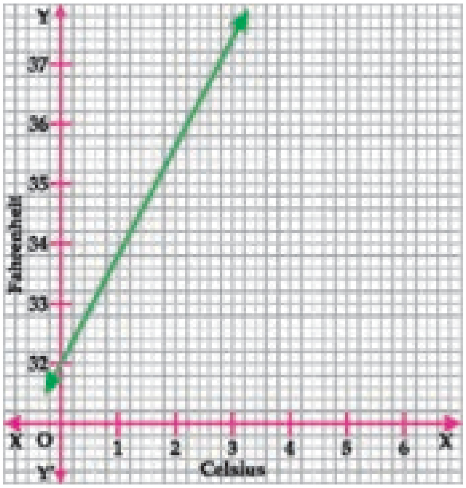
Example: Read the following conversion graph between Celsius and Fahrenheit to approximately convert:

i) 1° Celsius to Fahrenheit
ii) 3° Celsius to Fahrenheit
iii) 36° Fahrenheit to Celsius
iv) 37° Fahrenheit to Celsius

By using given scales.

Conversion graph between Celsius and Fahrenheit

Scale: 5 small squares = 1 Celsius along *x*-axis
5 small squares = 1 Fahrenheit along *y*-axis



Solution:

i) 1° Celsius to Fahrenheit

By reading the above graph as per given scale we can see

1° Celsius ≅ 33.8° Fahrenheit

ii) 3° Celsius to Fahrenheit

By reading the above graph as per given scale we can see

3° Celsius ≅ 37.4° Fahrenheit

iii) 36° Fahrenheit to Celsius

By reading the above graph as per given scale we can see

36° Fahrenheit ≅ 2.2° Celsius

iv) 37° Fahrenheit to Celsius

By reading the above graph as per given scale we can see

37° Fahrenheit ≅ 2.8° Celsius

(iv) Read the Currency Conversion Graph:

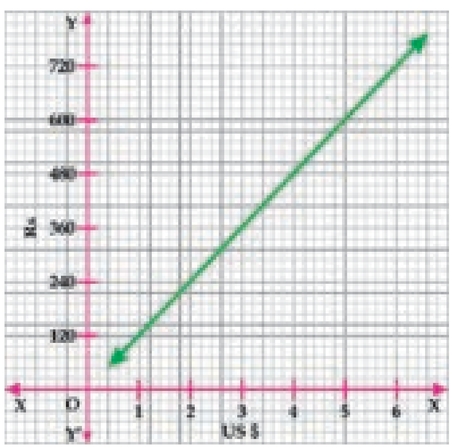
Example 1: Read the following conversion graph US dollar ($) and Pakistani rupees (Rs) to approximately convert:

i) 2 US $ to Rs
ii) 5 US $ to Rs
iii) 360 Rs to US $
iv) 720 Rs to US $

By using given scales.

Conversion graph between US dollar ($) to Pakistani rupees (Rs)

Scale: 5 small squares = 1 US $ along x-axis
5 small squares = 120 Rs along y-axis



Solution:

i) 2 US $ to Rs

By reading the above graph as per given scale we can see

2 $ ≅ Rs. 240

ii) 5 US \$ to Rs

By reading the above graph as per given scale we can see
5 \$ $\cong$ Rs. 600

iii) 360 Rs to US \$

By reading the above graph as per given scale we can see
Rs. 360 $\cong$ 3\$

iv) 720 Rs to US \$

By reading the above graph as per given scale we can see
Rs. 720 $\cong$ 6\$

7.3 Graphic Solution of Equations in Two Variables:

7.3.1 Solve simultaneous linear equation in two variables by graphical method

We have already studied the solution of two linear equations with two variables algebraically. Now in this section we will find the solution of the two linear equations in two variables graphically. The point of intersection of these two straight lines is the solution of these equations.

Example 01 Find the solution set graphically for the given equations.
$x + y = 3$ and $x - y = 5$.

Solution: Given equations are as under:

$x+y = 3$… (1)

and $x\text{-}y = 5$ … (2)

From equations (1) and (2), we can re-write them in term of y as under

$y = 3\text{-}x$… (3)

and $y = x\text{-}5$ … (4)

Now prepare separate tables for each linear equation

Table for the equation (3) is given below:

x	-1	0	1	2	3	4	…
y	4	3	2	1	0	-1	…

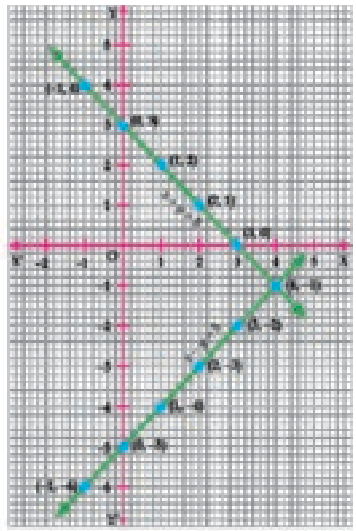
Table for the equation (4) given bellow:

x	-1	0	1	2	3	4	…
y	-6	-5	-4	-3	-2	-1	…

Now draw the straight lines by using the points of both the tables.

We see that graph of the given equations are straight lines l_1 and l_2, which meets at the point (4, -1), i.e., l_1 intersect l_2 at the point (4,-1).

Thus, solution Set is {(4,-1)}.



Example 02 Find the solution set graphically for the given equations. $y=2x+4$ and $y=2x-2$

Solution: The given equations are as under:

$$y = 2x + 4 \quad \dots \quad (1)$$

and $$y = 2x - 2 \quad \dots \quad (2)$$

Now make the separate sets for each linear equation.

For equation (1) table is given below:

x	-1	0	1	2	3	4
y	2	4	6	8	10	12

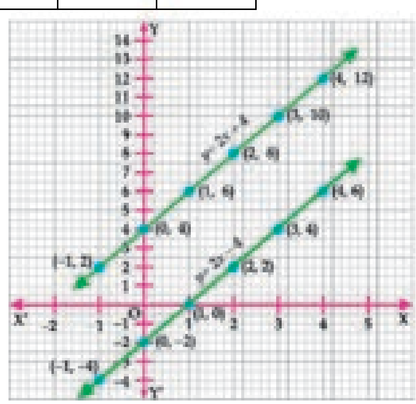
For equation (2) table is given below:

x	-1	0	1	2	3	4
y	-4	-2	0	2	4	6

Now locate these points for both the equations on the same graph and then make two lines by joining the points of the two equations.

We can see that straight lines are obtained from these equations which have no common point. It means these lines don't intersect at any point hence its solution set is empty.

Thus, solution set is { }.

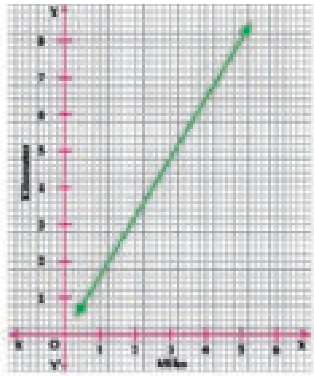


Exercise 7.3

1. Read the given conversion graph between miles and kilometer to approximately convert:

i) 1 miles to kilometers
ii) 3 miles to kilometers
iii) 2 kilometers to miles
iv) 8 kilometers to miles

By using given scales.



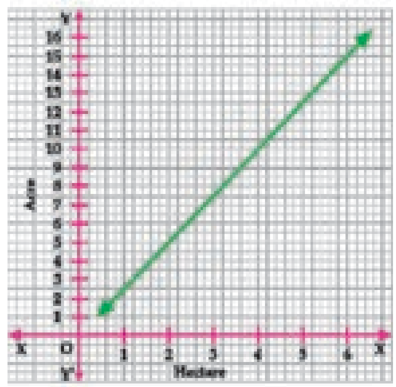
Conversion graph between miles and kilometer

Scale: 5 small squares = 1 mile along x-axis
5 small squares = 1 kilometer along y-axis

2. Read the given conversion graph between Hectares and Acres to approximately convert:

i) 2 hectares to acres
ii) 5 hectares to acres
iii) 5 acres to hectares
iv) 15 acres to hectares

By using given scales.



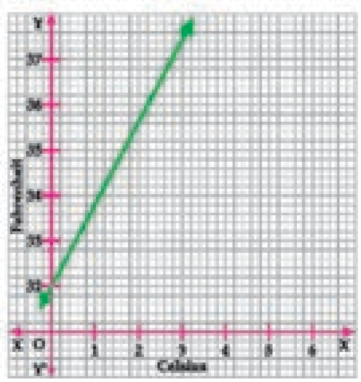
Conversion graph between hectares and acres

Scale: 5 small squares = 1 hectares along x-axis
2 small squares = 1 acres along y-axis

3. Read the given conversion graph between Celsius and Fahrenheit to approximately convert:

i) 2^{o} Celsius to Fahrenheit
ii) 1.80^{o} Celsius to Fahrenheit
iii) 32^{o} Fahrenheit to Celsius
iv) 36.4^{o} Fahrenheit to Celsius

By using given scales.



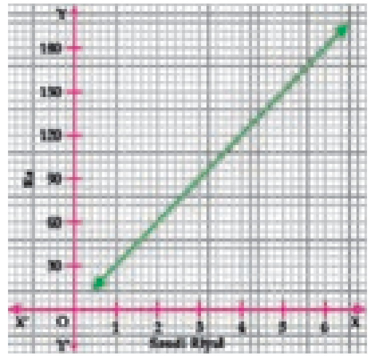
Conversion graph between Celsius and Fahrenheit

Scale: 5 small squares = 1 Celsius along x-axis
5 small squares = 1 Fahrenheit along y-axis

4. Read the given conversion graph Saudi Riyal and Pakistani rupees (Rs) to approximately convert:

i) 3 Saudi riyal to Rs
ii) 5.2 Saudi riyal to Rs
iii) 150 Rs to Saudi riyal
iv) 78 Rs to Saudi riyal

By using given scales.



Conversion graph between Saudi Riyal to Pakistani rupees (Rs)

Scale: 5 small squares = 1 Saudi riyal along x-axis
5 small squares = 30 Rs along y-axis

5. Solve the following simultaneous equations by graphical method.

i. $3x - 11 = y$; $x - 3y = 9$

ii. $x + y = 4$; $2x - 1 = 5y$

iii. $2x = y + 5$; $x = 2y + 1$

iv. $y = 3x - 5$; $x + y = 11$

v. $2x + y = 3$; $x - y = 0$

vi. $2x + 2 = y$; $y = x - 1$

vii. $5 = x + 4y$; $2x + 3y = 0$

viii. $3x = 5y - 2$; $3x + 5y = 8$

ix. $\frac{x+2}{5} + y = 6$; $y = 2x - 12$

x. $3x - 2y = 13$; $2x + 3y = 13$

Review Exercise 7

True and false question

1. Read the following sentences carefully and encircle T or F whichever is.

(i) The Cartesian plane is also called *xy*-plane. T / F

(ii) In 2nd quadrant both *x* and *y* coordinates are positive. T / F

(iii) The point (1, –2) lies in 1st quadrant. T / F

(iv) The (–3, –4) lies in the 4th quadrant. T / F

2. Tick (✓) the correct answer in the following:

(i) The point (–3, –4) is located in

(a) 1st quadrant (b) 2nd quadrant
(c) 3rd quadrant (d) 4th quadrant

(ii) The two coordinates axes intersect at an angle of

(a) 45^0 (b) 90^0
(c) 180^0 (d) 270^0

(iii) The line y = 4 is parallel to

(a) *x*-axis (b) *y*-axis
(c) Both axes (d) None

(iv) The line x=–5 is parallel to

(a) *x*-axis (b) *y*-axis
(c) Both axis (d) None

(v) The line x=–5 has a point on *x*-axis

(a) (–5, 5) (b) (0, -5)
(c) (–5, 0) (d) (5, 0)

(vi) The solution set of the line x=2 and x=5

(a) {(2, 5)} (b) {2, 5}
(c) {(0, 5)} (d) { }

(vii) The co-ordinate axes are mutually

(a) Perpendicular (b) Parallel
(c) Intersecting at 30^o (d) Intersecting at 45^o

Summary

- An ordered pair represents the position of a point in the Cartesian plane.
- The 2-dimensional Cartesian co-ordinate system is defined by two perpendicular lines i.e. x-axis and y-axis. Both the axes intersect each other at the specific point i.e., called origin (0, 0).
- Plane is divided into four quadrants by the axes.
- The Cartesian plane is also known as ***xy*-plane**.
- In quadrant I, both x and y-coordinates are positive i.e., $x > 0$ and $y > 0$.
- In quadrant II, x- co-ordinate (abscissa) is negative and y co-ordinate (ordinate) is positive i.e., $x < 0$ and $y > 0$.
- In quadrant III, both x and y- co-ordinate are negative i.e., $x<0$ and $y<0$.
- In quadrant IV, x- co-ordinate is positive and y- co-ordinate is negative i.e. $x > 0$ and $y < 0$.
- At the origin $x = y = 0$, so the origin has coordinates (0, 0).
- In general, any point in the Cartesian plane can be represented by the ordered pair (a, b), where 'a' is the x- co-ordinate (abscissa) and 'b' is the y- co-ordinate (ordinate).
- Graph of $x = a$ is parallel to the y-axes.
- Graph of $y = c$ is parallel to the x-axes.
- Graph of $y = mx$ always passes through the origin.
- Graph of $y = mx + c$ cut the y-axes at $y = c$.

Unit 8

• Weightage = 8%

QUADRATIC EQUATIONS

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Solve a quadratic equation in one variable by
  - Factorization,
  - Completing the squares.
- Use method of completing the squares to drive the quadratic formula.
- Use quadratic formula to solve quadratic equations.
- Solve equations, reducible to quadratic form, of the type $ax^4 + bx^2 + c = 0$, Quartic or Bi-quadratic equations.
- Solve the equations of the type $ap(x) + \frac{c}{p(x)} = b$, where a, b and c are rational numbers.
- Solve the reciprocal equations of the type $a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$, where a , b and c are rational numbers.
- Solve the exponential equations in which the variable occurs in exponents.
- Solve the equations of the type $(x+a)(x+b)(x+c)(x+d) = k$, where $a + b = c + d$ k $\neq 0$.
- Solve the equations of the type:
  - $\sqrt{(ax + b)} = cx + d$.
  - $\sqrt{(x + a)} + \sqrt{(x + b)} = \sqrt{(x + c)}$
  - $\sqrt{(x^2 + px + m)} + \sqrt{(x^2 + px + n)} = q$.

8.1 Quadratic Equations and their Solutions

8.1.1 Elucidate, then define Quadratic Equation in it Standard Form

A polynomial equation with degree 2 is called a quadratic equation.

The standard form of quadratic equation is $ax^2+bx+c=0$, where $a \neq 0$ and $a, b, c \in \mathbb{R}$. In this form a is the coefficient of x^2, b is the coefficient of x and c is the constant term.

In $ax^2+bx+c=0$, if $a=0$, then it reduces to linear equation i.e., $bx+c=0$ and if $b=0$ then it reduces to the pure quadratic form i.e., $ax^2+c=0$

Following are the examples of quadratic equations.

(i) $4x^2+4x+1=0$, (Quadratic equation is in the standard form)

(ii) $x^2-4=0$, (Pure quadratic equation)

8.1.2 Solve a quadratic equation in one variable by

- **Factorization**
- **Completing the square**

Here we consider two methods, for the solution of the quadratic equation.

(a) Method of factorization (b) Method of completing the square.

(a) Method of Factorization

Example 01 Solve: (i) $x^2+2x-15=0$ (ii) $2x^2-5x=12$

Solution (i): $x^2+2x-15=0$

$\Rightarrow \quad x^2+5x-3x-15=0$

$\Rightarrow \quad x(x+5)-3(x+5)=0$

$\Rightarrow \quad (x-3)(x+5)=0$

$\Rightarrow \quad x-3=0$ or $x+5=0$

$\Rightarrow \quad x=3$ or $x=-5$

Thus, S.S = $\{-5,3\}$

Solution (ii): $2x^2 - 5x = 12$

$$\Rightarrow \quad 2x^2 - 5x - 12 = 0$$

$$\Rightarrow \quad 2x^2 - 8x + 3x - 12 = 0$$

$$\Rightarrow \quad 2x(x-4) + 3(x-4) = 0$$

$$\Rightarrow \quad (x-4)(2x+3) = 0$$

$$\Rightarrow \quad x - 4 = 0 \quad \text{or} \quad 2x + 3 = 0$$

$$\Rightarrow \quad x = 4 \quad \text{or} \quad x = \frac{-3}{2}$$

Thus, the solution set is $\left\{-\frac{3}{2}, 4\right\}$

Example 02 Solve the pure quadratic equation $4m^2 - 1 = 0$ for m by factorization method:

Solution: $4m^2 - 1 = 0$

$$\Rightarrow \quad (2m)^2 - (1)^2 = 0$$

$$\Rightarrow \quad (2m-1)(2m-1) = 0 \quad \left[\ a^2 - b^2 = (a-b)(a+b)\right]$$

$$\Rightarrow \quad 2m - 1 = 0 \quad \text{or} \quad 2m + 1 = 0$$

i.e $\Rightarrow \quad 2m = 1 \quad \text{or} \quad 2m = -1$

$$\Rightarrow \quad m = \frac{1}{2} \quad \text{or} \quad m = -\frac{1}{2}$$

Thus, s.s = $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$.

(b) Method of Completing Square.

Method is explained as under:

(i) Write the equation in the standard form i.e. $ax^2+bx+c = 0$

(ii) Divide both the sides of the equation by leading coefficient of x^2 in order to make it 1.

(iii) Shift the constant term to the R.H.S.

(iv) For completing the square add $\left(\frac{\text{coefficient of } x}{2}\right)^2$ on both sides

(v) Write the L.H.S. of the equations as a perfect square and then simplify the R.H.S.

(vi) Take the square root of both the sides of the given equation. Solve the resulting equation to find the solution of the equation and then write the solution set.

Example 01 Solve $2x^2+8x-1=0$

Solution: $2x^2+8x-1=0$

$\Rightarrow \quad 2x^2+8x=1 \quad ...(i)$

$\Rightarrow \quad x^2+4x=\frac{1}{2} \quad ...(ii)$ [By dividing equation (i) by 2]

By adding $\left[\frac{1}{2}\times 4\right]^2=4$ on both the sides in equation (ii)

we get,

$$x^2+4x+4=\frac{1}{2}+4$$

$$\Rightarrow x^2+2(2)x+(2)^2=\frac{1}{2}+(2)^2$$

$$\Rightarrow (x+2)^2=\frac{1}{2}+4$$

$$\Rightarrow (x+2)^2=\frac{9}{2}$$

$$\Rightarrow x+2=\pm\frac{3}{\sqrt{2}}$$

$\Rightarrow x+2=\frac{3}{\sqrt{2}}$	$x+2=-\frac{3}{\sqrt{2}}$
$\Rightarrow x+2=\frac{3\sqrt{2}}{2}$	$x+2=\frac{-3\sqrt{2}}{2}$
$\Rightarrow x=-2+\frac{3\sqrt{2}}{2}$	$x=-2-\frac{3\sqrt{2}}{2}$
$\Rightarrow x=\frac{-4+3\sqrt{2}}{2}$	$x=\frac{-4-3\sqrt{2}}{2}$

Thus, s.s is $\left\{\frac{-4+3\sqrt{2}}{2}, \frac{-4-3\sqrt{2}}{2}\right\}$.

Exercise 8.1

1. **Solve the following quadratic equations by factorization method:**

   (i) $x^2+5x+6=0$ (ii) $6x^2-x-1=0$ (iii) $x^2-11x+30=0$

   (iv) $x^2-2x=0$ (v) $x^2-2x-15=0$ (vi) $12x^2-41x+24=0$

   (vii) $(x-5)^2-9=0$ (viii) $(3x+4)^2-16=0$

2. **Solve each of the following by completing the square method:**

   (i) $x^2+6x+1=0$ (ii) $(3x+2)(x+2)=6-2(x+1)$. (iii) $3x^2-8x=-1$

   (iv) $24x^2=-10x+21$ (v) $2(x^2-3)-3x=2(x+3)$ (vi) $2x^2+4x-1=0$

3. **The equation $3x^2+bx-8=0$ has 2 as one of its roots.**

   (i) What is the value of b?

   (ii) What is the other root of the equation?

8.2 Quadratic Formula

For equation $ax^2+bx+c=0, a\neq 0$ we use the following formula to solve it i.e. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ the formula is known as Quadratic formula.

8.2.1 Use method of completing the square to derive the Quadratic Formula.

The standard form of a quadratic equation is given by

$$ax^2+bx+c=0 \quad \text{where } a\neq 0 \qquad \text{..........(i)}$$

By dividing a on both sides of equation (i), we get

$$\frac{ax^2}{a}+\frac{bx}{a}+\frac{c}{a}=\frac{0}{a}$$

$$\therefore \quad x^2+\frac{bx}{a}+\frac{c}{a}=0$$

By shifting constant term $\frac{c}{a}$ to R.H.S

$$\therefore \quad x^2+\frac{bx}{a}=-\frac{c}{a} \quad ...\text{(ii)}$$

By adding $\left(\frac{b}{2a}\right)^2$ on both sides of equation (ii)

$$\therefore \quad x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\Rightarrow \quad x^2+2(x)\left(\frac{b}{2a}\right)+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{-4ac+b^2}{4a^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad \boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$ This is known as Quadratic Formula.

8.2.2 Use of Quadratic Formula to solve Quadratic Equations.

Example 01 Solve by using quadratic formula

(i) $2x^2-5x-3=0$ (ii) $x^2+x+1=0$

Solution (i): $2x^2-5x-3=0$

Here, $a=2, b=-5$ and $c=-3$

By using quadratic formula

i.e $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-3)}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm\sqrt{25-(-24)}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm\sqrt{49}}{4}$$

$$\Rightarrow \quad x=\frac{5\pm7}{4}$$

$\Rightarrow \quad x = \frac{5+7}{4}$ | $x = \frac{5-7}{4}$

$\Rightarrow \quad x = \frac{12}{4}$ | $x = \frac{-2}{4}$

$\Rightarrow \quad x = 3$ | $x = -\frac{1}{2}$

so, the roots are 3 and $-\frac{1}{2}$

Thus, $S.S = \left\{3, -\frac{1}{2}\right\}$.

Solution (ii): $x^2 + x + 1 = 0$

Here, $a = 1$, $b = 1$ and $c = 1$

By using quadratic formula

$$\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2},$$

Thus, $S.S = \left\{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$

Exercise 8.2

Solve the following equations by using the Quadratic Formula:

(i) $x^2 - 2x = 15$ (ii) $10x^2 + 19x - 15 = 0$ (iii) $x^2 = -x + 1$

(iv) $2x = 9 - 3x^2$ (v) $9x^2 = 12x - 49$ (vi) $\frac{1}{2}x^2 + \frac{3}{4}x - 1 = 0$

(vii) $3x^2 - 2x + 2 = 0$ (viii) $6x^2 - x - 1 = 0$ (ix) $4x^2 - 10x = 0$

(x) $x^2 - 1 = 0$ (xi) $x^2 - 6x + 9 = 0$ (xii) $\frac{1}{x+4} - \frac{1}{x-4} = 4$

8.3 Equations Reducible to Quadratic Form

There are various types of equations which are not quadratic, but can be reduced into the quadratic form by taking suitable substitution.

8.3.1 Solve equation reducible to quadratic form of the type $ax^4 + bx^2 + c = 0, a \neq 0$ i.e., quartic or bi-quadratic equation.

Consider the equation $ax^4 + bx^2 + c = 0$, it is quartic or bi-quadratic equation as it has degree 4, and can be reduced into the quadratic equation having form $ay^2 + by + c = 0$, where $y = x^2$. The method is explained by the following example.

Example Solve the quartic equation $4x^4 - 25x^2 + 36 = 0$

Solution: $4x^4 - 25x^2 + 36 = 0$ . . . (i)

This equation can be written as:

$$4(x^2)^2 - 25(x^2) + 36 = 0 \quad \ldots \text{(ii)}$$

By putting $y = x^2$ in equation (ii), we have,

$$4y^2 - 25y + 36 = 0$$

Here, $a = 4, b = -25$ and $c = 36$

$$\because \quad y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \quad y = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(36)}}{2(4)}$$

$$y = \frac{25 \pm \sqrt{625 - 576}}{2(4)} = \frac{25 \pm \sqrt{49}}{8} = \frac{25 \pm 7}{8}$$

i.e., $y = \dfrac{25 + 7}{8}$ | $y = \dfrac{25 - 7}{8}$

$y = \dfrac{32}{8} = 4$ | $y = \dfrac{18}{8} = \dfrac{9}{4}$

but, $y = x^2$, then

$x^2 = 4$ | $x^2 = \dfrac{9}{4}$

$x = \pm 2$ | $x = \pm \frac{3}{2}$

Thus, $S.S = \left\{\pm 2,\ \pm\frac{3}{2}\right\}$

8.3.2 Solve equation of the type $ap(x)+\frac{b}{p(x)}=c$ where a,b and c are real numbers, a≠0, where $p(x)$ is an algebraic expression

Example 01 Solve $8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2$

Solution: $8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2$ . . . (i)

Let $y = \sqrt{x+3} \Rightarrow \frac{1}{y} = \frac{1}{\sqrt{x+3}}$, so (i) becomes

$\Rightarrow \quad 8y - \frac{1}{y} = 2$

$\Rightarrow \quad 8y^2 - 1 = 2y$

$\Rightarrow \quad 8y^2 - 2y - 1 = 0$

$\Rightarrow \quad 8y^2 - 4y + 2y - 1 = 0$

$\Rightarrow \quad 4y(2y-1) + 1(2y-1) = 0$

$\Rightarrow \quad (4y+1)(2y-1) = 0$

$\Rightarrow \quad 4y+1=0$ | $2y-1=0$

$\Rightarrow \quad y = \frac{-1}{4}$ | $y = \frac{1}{2}$

when $y = -\frac{1}{4}$, | when $y = \frac{1}{2}$

$\therefore \quad \sqrt{x+3} = -\frac{1}{4}$ | $\therefore \quad \sqrt{x+3} = \frac{1}{2}$

i.e Squaring on both sides, we get | i.e Squaring on both sides, we get

$\Rightarrow \quad x+3 = \frac{1}{16}$ | $x+3 = \frac{1}{4}$

$\Rightarrow \quad x = \frac{1}{16} - 3$	$x = \frac{1}{4} - 3$
$\Rightarrow \quad x = \frac{1-48}{16}$	$x = \frac{1-12}{4}$
$\Rightarrow \quad x = -\frac{47}{16}$	$x = -\frac{11}{4}$

As eq-(i) is a radical equation. So verification of roots of (i) is essential.

Verification: $$8\sqrt{x+3} - \frac{1}{\sqrt{x+3}} = 2$$

By putting $x = -\frac{47}{16}$ in eq....(i)

$$8\sqrt{\frac{-47}{16}+3} - \frac{1}{\sqrt{\frac{-47}{16}+3}} = 2$$

$$8\sqrt{\frac{-47+48}{16}} - \frac{1}{\sqrt{\frac{-47+48}{16}}} = 2$$

$$8\sqrt{\frac{1}{16}} - \frac{1}{\sqrt{\frac{1}{16}}} = 2$$

$$8\left(\frac{1}{4}\right) - \frac{1}{\left(\frac{1}{4}\right)} = 2$$

$$2 - 4 = 2$$

$$-2 \neq 2$$

Not verified

By putting $x = -\frac{11}{4}$ in eq....(i)

$$8\sqrt{\frac{-11}{4}+3} - \frac{1}{\sqrt{\frac{-11}{4}+3}} = 2$$

$$8\sqrt{\frac{-11+12}{4}} - \frac{1}{\sqrt{\frac{-11+12}{4}}} = 2$$

$$8\sqrt{\frac{1}{4}} - \frac{1}{\sqrt{\frac{1}{4}}} = 2$$

$$8\left(\frac{1}{2}\right) - \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$4 - 2 = 2$$

$$2 = 2$$

Verified

On verification it is found that $x = -\frac{47}{16}$ does not satisfy the original equation.

Hence, it is an extraneous root, and cannot be included in the solution set.

Thus, S.S = $\left\{-\frac{11}{4}\right\}$.

8.3.3 Solve the Reciprocal Equation of the type $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, where a,b and c are rational numbers.

Definition:

An equation in x is said to be a reciprocal equation, if it remains un-changed when x is replaced by $\frac{1}{x}$.

The method for solving reciprocal equation of the type $a\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0$, where a, b, c are rational numbers, explained through an example.

Example 01 Solve: $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0$

Solution: $2\left(x^2+\frac{1}{x^2}\right)-9\left(x+\frac{1}{x}\right)+14=0 \quad . . . \text{(i)}$

Let $x+\frac{1}{x}=y$ then $x^2+\frac{1}{x^2}=y^2-2$, so, equation (i) becomes

$$2\left(y^2-2\right)-9y+14=0$$

$$\Rightarrow \quad 2y^2-9y+10=0$$

$$\Rightarrow \quad 2y^2-4y-5y+10=0$$

$$\Rightarrow \quad 2y(y-2)-5(y-2)=0$$

$$\Rightarrow \quad (y-2)(2y-5)=0$$

i.e. $y-2=0$	$2y-5=0$
$\Rightarrow \quad y=2$	$y=\frac{5}{2}$
when $y=2$	when $y=\frac{5}{2}$
$\therefore \quad x+\frac{1}{x}=2$	$\therefore \quad x+\frac{1}{x}=\frac{5}{2}$
$\Rightarrow \quad x^2+1=2x$	$\Rightarrow \quad \frac{x^2+1}{x}=\frac{5}{2}$
$\Rightarrow \quad x^2-2x+1=0$	$\Rightarrow \quad 2x^2+2=5x$

$\Rightarrow \quad (x-1)^2 = 0$

$\Rightarrow \quad x-1 = 0$

$\Rightarrow \quad x = 1$

$\Rightarrow \quad 2x^2 - 5x + 2 = 0$

$\Rightarrow \quad 2x^2 - 4x - x + 2 = 0$

$\Rightarrow \quad 2x(x-2) - 1(x-2) = 0$

$\Rightarrow \quad (x-2)(2x-1) = 0$

Either $x - 2 = 0$

$\Rightarrow \quad x = 2$

or $2x - 1 = 0$

$\Rightarrow \quad x = \frac{1}{2}$

Thus, S.S $= \left\{\frac{1}{2}, 2, 1\right\}$.

8.3.4 Solve the Exponential Equations

Definition:

An equation in which the variable appears as an exponent, is called an exponential equation. Solution of such type of equation is explained through an example.

Example 01 solve $7^{1+x} + 7^{1-x} = 50$

Solution: $7^{1+x} + 7^{1-x} = 50$ . . . (i)

$7.7^x + 7.7^{-x} - 50 = 0$ (Splitting power)

$\Rightarrow \quad 7.7^x + \frac{7}{7^x} = 50$ . . . (ii)

Let $y = 7^x$

$\therefore$ above equation (ii) reduces as under,

$7y + \frac{7}{y} - 50 = 0$,

$\Rightarrow \quad 7y^2 + 7 - 50y = 0$

$\Rightarrow \quad 7y^2 - 50y + 7 = 0$

$\Rightarrow \quad 7y^2 - 49y - y + 7 = 0$ (Factorizing)

$\Rightarrow \quad 7y(y-7) - 1(y-7) = 0$

$\Rightarrow \quad (7y-1)(y-7) = 0$

Either,

$7y-1=0$	or $y-7=0$
$\Rightarrow \quad y=\frac{1}{7}$	$\Rightarrow \quad y=7$
when $y=\frac{1}{7} \Rightarrow 7^x=\frac{1}{7}=7^{-1}$	when $y=7 \Rightarrow 7^x=7^1$
$\Rightarrow \quad x=-1$	$x=1$

Thus, S.S $=\{-1,1\}$.

8.3.5 Solve the Equations of the type $(x+a)(x+b)(x+c)(x+d)=k$, where, $a+b=c+d$ and the constant $k\neq 0$.

Example 01 $(x+1)(x+2)(x+3)(x+4)=48$

Solution: $(x+1)(x+2)(x+3)(x+4)=48$

By re-arranging the factors, we have

$(x+1)(x+4)(x+2)(x+3)=48$

$\left(x^2+4x+x+4\right)\left(x^2+2x+3x+6\right)=48$

$\left(x^2+5x+4\right)\left(x^2+5x+6\right)=48 \ldots$ (i)

Let $x^2+5x=t$ ...(ii)

By substituting in equation (i)

$\Rightarrow \quad (t+4)(t+6)=48$

$\Rightarrow \quad t^2+4t+6t+24=48$

$\Rightarrow \quad t^2+10t-24=0$

$\Rightarrow \quad t^2+12t-2t-24=0$

$\Rightarrow \quad t(t+12)-2(t+12)=0$

$(t+12)(t-2)=0$

Either,

$t+12=0$	or $t-2=0$
$\Rightarrow \quad t=-12$	$t=2$
Substituting in equation (ii)	Substituting in equation (ii)
$\Rightarrow \quad x^2+5x=-12$	$x^2+5x=2$
$\Rightarrow \quad x^2+5x+12=0$	$x^2+5x-2=0$

Here: $a=1$, $b=5$ and $c=12$,

$$\Rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=\frac{-5\pm\sqrt{25-48}}{2}$$

$$\Rightarrow \quad x=\frac{-5\pm\sqrt{-23}}{2}$$

$$\Rightarrow \quad x=\frac{-5\pm i\sqrt{23}}{2}$$

Here: $a=1$, $b=5$ and $c=-2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-5\pm\sqrt{25+8}}{2}$$

$$x=\frac{-5\pm\sqrt{33}}{2}$$

$$\text{S.S}=\left\{\frac{-5\pm i\sqrt{23}}{2},\frac{-5\pm\sqrt{33}}{2}\right\}$$

Exercise 8.3

Solve the following equations:

1. $x^4-8x^2-9=0$
2. $x^4-3x^2-4=0$
3. $12x^4-11x^2+2=0$
4. $\frac{2x+3}{x+1}+6\left(\frac{x+1}{2x+3}\right)=7$
5. $\sqrt{\frac{2x^2+1}{x^2+1}}+6\sqrt{\frac{x^2+1}{2x^2+1}}=5$
6. $5^{x+1}+5^{2-x}=5^3+1$
7. $2^x+\frac{16}{2^x}=8$
8. $2\left(\frac{x}{x+1}\right)^2-5\left(\frac{x}{x+1}\right)+2=0$
9. $4\left(\frac{x}{x-1}\right)^2-4\left(\frac{x}{x-1}\right)+1=0$
10. $9^{x+2}-6\cdot 3^{x+1}+1=0$
11. $2^x+2^{-x+6}-20=0$
12. $(x-1)(x+5)(x+8)(x+2)=880$
13. $(x+1)(x+2)(x+3)(x+4)=120$
14. $(x-2)(x+1)(x+3)(x-4)=24$

8.4 Radical Equations

Definition

An equation in which the variable appears under the radical sign, is called a radical equation.

Solution of the radical equations must be verified as it may have extraneous root.

8.4.1 Solution of the equations of the type:

Type (i) $\sqrt{ax+b} = cx + d$ Type (ii) $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Type (iii) $\sqrt{x^2+px+m} + \sqrt{x^2+px+n} = q$

Type (i): $\sqrt{ax+b} = cx + d$

Example 01 **Solve** $\sqrt{217-x} = x-7$

Solution: $\sqrt{217-x} = x-7$

Squaring on both sides, we have,

$(\sqrt{217-x})^2 = (x-7)^2$

$\Rightarrow \quad 217-x=(x-7)^2$

$\Rightarrow \quad 217-x=x^2-14x+49$

$\Rightarrow \quad x^2-13x-168=0$

$\Rightarrow \quad x^2-21x+8x-168=0$

$\Rightarrow \quad x(x-21)+8(x-21)=0$

$\Rightarrow \quad (x-21)(x+8)=0$

Either, $x=21$ *or* $x=-8$

verification : when $x=21$

$\sqrt{217-x} = x-7$

$\therefore \quad \sqrt{217-21} = 21-7$

$\Rightarrow \quad \sqrt{196} = 14$

$\Rightarrow \quad 14 = 14$

verified

verification : when $x=-8$

$\sqrt{217-x} = x-7$

$\therefore \quad \sqrt{217-(-8)} = -8-7$

$\Rightarrow \quad \sqrt{225} = -15$

$\Rightarrow \quad 15 \neq -15$

not verified

On verification it is found that $x=-8$ does not satisfy the original equation, hence it is an extraneous root, and can't be included in the solution set. Therefore, S.S = {21}

Type (ii) : $\sqrt{x+a} + \sqrt{x+b} = \sqrt{x+c}$

Example 01 **Solve:** $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Solution: $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Squaring on both sides, we get,

$(\sqrt{x+7} + \sqrt{x+2})^2 = (\sqrt{6x+13})^2$

$\Rightarrow (\sqrt{x+7})^2 + 2(\sqrt{x+7})(\sqrt{x+2}) + (\sqrt{x+2})^2 = 6x+13$

$\Rightarrow x+7 + 2\sqrt{(x+7)(x+2x)} + x+2 = 6x+13,$

$\Rightarrow 2x+9+2\sqrt{x^2+7x+2x+14} = 6x+13$

$\Rightarrow 2\sqrt{x^2+9x+14} = 4x+4$

$\Rightarrow \sqrt{x^2+9x+14} = 2x+2$

$\Rightarrow \sqrt{x^2+9x+14} = 2(x+1)$

Again Squaring on both sides

$\left(\sqrt{x^2+9x+14}\right)^2 = [2(x+1)]^2$

$\Rightarrow x^2+9x+14 = 4(x+1)^2$

$\Rightarrow x^2+9x+14 = 4(x^2+2x+1)$

$\Rightarrow x^2+9x+14 = 4x^2+8x+4$

$\Rightarrow 3x^2-x-10 = 0$

$\Rightarrow 3x^2+5x-6x-10 = 0$

$\Rightarrow x(3x+5)-2(3x+5) = 0$

$\Rightarrow (x-2)(3x+5) = 0$

Either

$3x+5 = 0$

$\Rightarrow x = -\frac{5}{3}$

Verification:

when $x = -\frac{5}{3}$

$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

or $x-2 = 0$

$\Rightarrow x=2$

Verification

when $x=2$

$\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

$$\sqrt{-\frac{5}{3}+7}+\sqrt{-\frac{5}{3}+2}=\sqrt{6\left(-\frac{5}{3}\right)+13}$$

$$\Rightarrow \quad \sqrt{\frac{16}{3}}+\sqrt{\frac{1}{3}}=\sqrt{3}$$

$$\Rightarrow \quad \frac{4}{\sqrt{3}}+\frac{1}{\sqrt{3}}=\sqrt{3}$$

$$\Rightarrow \quad \frac{5}{\sqrt{3}}\neq\sqrt{3}$$

Not verified.

$$\sqrt{2+7}+\sqrt{2+2}=\sqrt{6(2)+13},$$

$$\Rightarrow \quad \sqrt{9}+\sqrt{4}=\sqrt{25}$$

$$\Rightarrow \quad 3+2=5$$

$$\Rightarrow \quad 5=5$$

Verified.

Since $-\frac{5}{3}$ is an extraneous root, therefore the solution is $x=2$

Thus, the solution set = {2}.

Type (iii): $\sqrt{x^2+px+m}+\sqrt{x^2+px+n}=q$

Example 01 Solve: $\sqrt{x^2-3x+21}-\sqrt{x^2-3x+5}=2$

Solution: Put $y=x^2-3x$ in the given equation, we have,

$$\sqrt{x^2-3x+21}-\sqrt{x^2-3x+5}=2$$

$$\sqrt{y+21}-\sqrt{y+5}=2$$

$$\sqrt{y+21}=2+\sqrt{y+5}$$

Squaring on both sides, we get

$$(\sqrt{y+21})^2=(2+\sqrt{y+5})^2$$

$$\Rightarrow \quad y+21=(2)^2+4\sqrt{y+5}+\left(\sqrt{y+5}\right)^2$$

$$\Rightarrow \quad y+21=4+4\sqrt{y+5}+y+5$$

$$\Rightarrow \quad 4\sqrt{y+5}=y+21-4-y-5$$

$$\Rightarrow \quad 4\sqrt{y+5}=12$$

$$\Rightarrow \quad \sqrt{y+5}=3$$

Again squaring on both the sides, we have,

$$\Rightarrow \quad y+5=9$$

$$\Rightarrow \quad y=4$$

Put $y=4$ in the substitution $y=x^2-3x$, we have,

$4 = x^2 - 3x$

$\Rightarrow \quad x^2 - 3x - 4 = 0$

$\Rightarrow \quad x^2 - 4x + x - 4 = 0$

$\Rightarrow \quad x(x-4) + 1(x-4) = 0$

$\Rightarrow \quad (x-4)(x+1) = 0$

Either $x - 4 = 0$

$\Rightarrow \quad x = 4$

For $x = 4$

$\sqrt{x^2 - 3x + 21} - \sqrt{x^2 - 3x + 5} = 2$

$\sqrt{(4)^2 - 3(4) + 21} - \sqrt{(4)^2 - 3(4) + 5} = 2$

$\sqrt{16 - 12 + 21} - \sqrt{16 - 12 + 5} = 2$

$\sqrt{25} - \sqrt{9} = 2$

$5 - 3 = 2$

$2 = 2$

or $\quad x + 1 = 0$

$x = -1$

Verification :

For $x = -1$

$\sqrt{x^2 - 3x + 21} - \sqrt{x^2 - 3x + 5} = 2$

$\sqrt{(-1)^2 - 3(-1) + 21} - \sqrt{(-1)^2 - 3(-1) + 5} = 2$

$\sqrt{1 + 3 + 21} - \sqrt{1 + 3 + 5} = 2$

$\sqrt{25} - \sqrt{9} = 2$

$5 - 3 = 2$

$2 = 2$

Hence both roots are satisfied by the given equation. Thus, the solution set is $\{-1, 4\}$.

Exercise 8.4

Solve the following equations :

1. $x + \sqrt{x+5} = 7$
2. $\sqrt{x-2} = 8 - x$
3. $\sqrt{7-5x} + \sqrt{13-5x} = 3\sqrt{4-2x}$
4. $\sqrt{x+2} + \sqrt{x+7} = \sqrt{6x+13}$
5. $\sqrt{2x^2 + 3x + 4} + \sqrt{2x^2 + 3x + 9} = 5$
6. $\sqrt{y^2 - 3y + 9} - \sqrt{y^2 - 3y + 36} + 3 = 0$

Review Exercise 8

1. Fill in the blanks

(i) A polynomial equation in which degree of variable is ______________ called quadratic equation.

(ii) Standard form of quadratic equation is ______________ .

(iii) $3^x + 3^{2x} = 1$ is called ______________ equation.

(iv) Solution of $3^x = 9$ is______________.

(v) Solution of $ax^2 + bx + c = 0$ is ______________ .

2. Tick (✓) the correct answer

(i) Degree of quadratic equation is

(a) 1 (b) 2 (c) 3 (d) 4

(ii) Standard form of quadratic equation is

(a) $ax^2 + bx + c = 0, a \neq 0$ (b) $ax^2 + c = 0, a \neq 0$

(c) $ax^2 + bx = 0, a \neq 0$ (d) $ax^3 + bx^2 + c = 0, a \neq 0$

(iii) The Quadratic Formula for $ax^2 + bx + c = 0, a \neq 0$ is

(a) $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2}$ (b) $x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

(c) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (d) $x = \dfrac{-b \pm \sqrt{b^2 + 4ac}}{2a}$

(iv) Solution set of $x^2 + 10x + 24 = 0$ is

(a) $\{-6, -4\}$ (b) $\{-6, 4\}$

(c) $\{6, 4\}$ (d) $\{6, -4\}$

(v) How many maximum roots of quadratic equation are

(a) 2 (b) 3 (c) 1 (d) 4

(vi) Two linear factors of $x^2 - 15x + 56$ are

(a) $(x-7)$ and $(x + 8)$ (b) $(x+7)$ and $(x-8)$

(c) $(x-7)$ and $(x-8)$ (d) $(x+7)$ and $(x + 8)$

(vii) Polynomial equation, which remains unchanged when x is replaced by $\dfrac{1}{x}$ is called a/ an

(a) Exponential equation (b) Reciprocal equation

(c) Radical equation (d) none of these

(viii) An equation of the type of $3^x + 3^{2-x} + 6 = 0$ is a/an

(a) Exponential equation (b) Radical equation
(c) Reciprocal equation (d) none of these

(ix) The solution set of equation $4x^2 - 16 = 0$ is
(a) $\{\pm 4\}$ (b) $\{4\}$ (c) $\{\pm 2\}$ (d) none of these

(x) An equation of the form $2x^4 - 3x^3 + 7x^2 - 3x + 2 = 0$ is called a/an

(a) Reciprocal equation (b) Radical equation
(c) Exponential equation (d) none of these

3. True and false questions

Read the following sentences carefully and en-circle 'T' in case of true and 'F' in case of false statement.

(i) Every quadratic equation can be solved by factorization. T/F
(ii) Every Quartic equation has two roots. T/F
(iii) Every Quadratic equation can have no solution. T/F
(iv) $ax^2 + bx + c = 0$ is called the quadratic equation in x if a=0 and b, c are real numbers. T/F
(v) Extraneous root satisfy the equation. T/F
(vi) Extraneous roots do not satisfy the equation. T/F
(vii) In the quadratic equation the highest exponent of the variable is two. T/F

Summary

- A Polynomial equation in which degree of a variable is 2, called quadratic equation.
- $ax^2 + bx + c = 0$, $a \neq 0$, a, b, c are real numbers is called standard form of a quadratic equation.
- Formula for quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- In exponential equations, variables occur in exponents.
- An equation in which the variable appears under the radical sign is called a radical equation.

Unit 9

• Weightage = 7%

CONGRUENT TRIANGLES

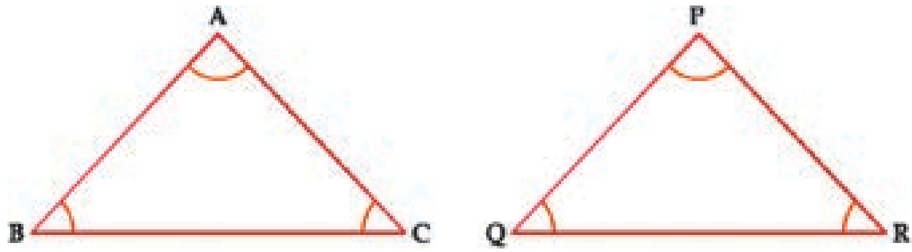
Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
- In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding sides and angles of the other, the two triangles are congruent.
- If two angles of a triangle are congruent then the sides opposite to them are also congruent.
- In the correspondence of the two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, the two triangles are congruent or similar triangles.
- If in the correspondence of two right angled triangles, the hypotenuse and one side of one are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent.

Introduction

A triangle has six elements, three sides and three angles. If we are given two triangles ABC and PQR, we can associate their vertices to establish a (1-1) correspondence between the sides and angles of these triangles in six different ways given as under:



In the correspondence $\Delta ABC \leftrightarrow \Delta PQR$ it means

(i)	$\angle A \leftrightarrow \angle P$	($\angle A$ corresponds to $\angle P$).
(ii)	$\angle B \leftrightarrow \angle Q$	($\angle B$ corresponds to $\angle Q$).
(iii)	$\angle C \leftrightarrow \angle R$	($\angle C$ corresponds to $\angle R$).
(iv)	$\overline{AB} \leftrightarrow \overline{PQ}$	($\overline{AB}$ corresponds to $\overline{PQ}$).
(v)	$\overline{BC} \leftrightarrow \overline{QR}$	($\overline{BC}$ corresponds to $\overline{QR}$).
(vi)	$\overline{CA} \leftrightarrow \overline{RP}$	($\overline{CA}$ corresponds to $\overline{RP}$).

9.1 Congruent triangles

"Sameness of size and shape" in the mathematics called congruence.

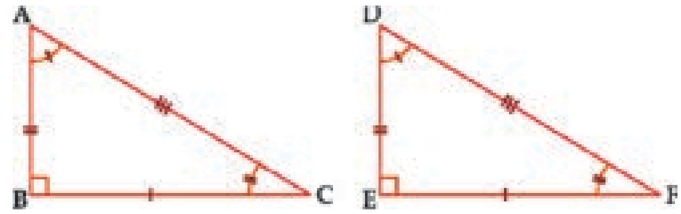
Consider two cars having different colours and positions as shown in the adjacent figure. But they have same size and shape. These two cars are said to be congruent. If we keep the picture of one car on the other car then they will overlap with each other.

ACTIVITY

Exploration

Can you identify any congruent figures or objects in your classroom or school? Make a list of these congruent figures by drawing or taking photos.

Two triangles are said to be congruent if their corresponding angles and sides are congruent.
Let's see the figures.



These two triangles ABC and DEF are congruent and written as:

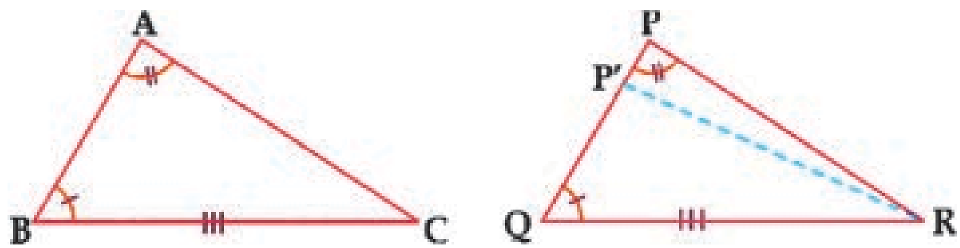
$$\Delta ABC \cong \Delta DEF$$

The ΔABC and ΔDEF having their corresponding sides and angles are equal in measure.
Note: Following results are useful.

(i) Identity congruence i.e $\Delta ABC \cong \Delta ABC$.
(ii) Symmetric property i.e $\Delta ABC \cong \Delta PQR$ then $\Delta PQR \cong \Delta ABC$.
(iii) Transitive property of congruence, if
$\Delta ABC \cong \Delta PQR$ and $\Delta PQR \cong \Delta DEF$, then $\Delta ABC \cong \Delta DEF$.

Theorem 9.1.1 (A.S.A. ≅ A.S.A.)

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding side and angles of the other, the two triangles are congruent.



Given:

In $\Delta ABC \leftrightarrow \Delta PQR$, then
$\angle B \cong \angle Q$, $m\overline{BC} \cong m\overline{QR}$,
and $\angle A \cong \angle P$.

To prove:

$\Delta ABC \cong \Delta PQR$

Construction:

Suppose, $\overline{AB} \ncong \overline{PQ}$ then take a point P′ on $\overline{PQ}$ such that $\overline{AB} \cong \overline{P'Q}$.
Join P′ to R.

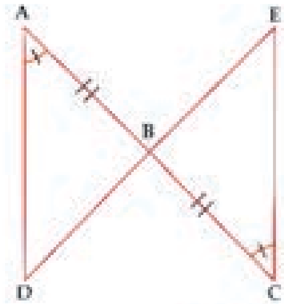
Proof:

Statements	Reasons
1. In $\Delta ABC \leftrightarrow \Delta PQR$	**1.** Correspondence of two Δs
i. $\angle A \cong \angle P$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
2. $\therefore$ $\angle C \cong \angle R$	**2.** Two corresponding angles of both triangles are congruent.
3. If $\overline{BA} \ncong \overline{QP}$, take a point P′ on $\overline{QP}$ (or $\overline{QP}$ produced) such that: $\overline{QP'} \cong \overline{BA}$	**3.** Assumption
4. In $\Delta ABC \leftrightarrow \Delta P'QR$	**4.** Correspondence of two Δs
i. $\overline{BC} \cong \overline{QR}$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
iii. $\overline{BA} \cong \overline{QP'}$	iii. By supposition
5. $\therefore$ $\Delta ABC \cong \Delta P'QR$	**5.** S.A.S postulate
6. $\therefore$ $\angle C \cong \angle QRP'$	**6.** Corresponding $\angle$s of congruent Δs.
7. But $\angle C \cong \angle QRP$	**7.** Proved in 2 (above).
8. $\therefore$ $\angle QRP' \cong \angle QRP$	**8.** Transitive property of congruence
9. This is possible only when points P′ and P coincide and $\overline{RP'} \cong \overline{RP}$	**9.** By angle construction postulate
10. Hence $\overline{BA} \cong \overline{QP}$	**10.** As P and P′ coincide.
11. In $\Delta ABC \leftrightarrow \Delta PQR$	**11.** Correspondence of two Δs
i. $\overline{BC} \cong \overline{QR}$	i. Given
ii. $\angle B \cong \angle Q$	ii. Given
iii. $\overline{BA} \cong \overline{QP}$	iii. Proved above
12. $\therefore$ $\Delta ABC \cong \Delta PQR$	**12.** S.A.S Postulate.

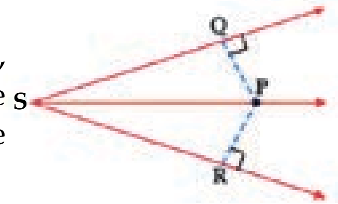
Q.E.D.

Exercise 9.1

1. In the adjacent figure, $m\overline{AB} = m\overline{CB}$ and $\angle A \cong \angle C$ prove that $\Delta ABD \cong \Delta CBE$

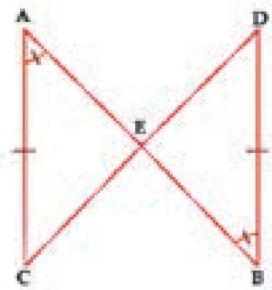


2. From a point on the line bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.

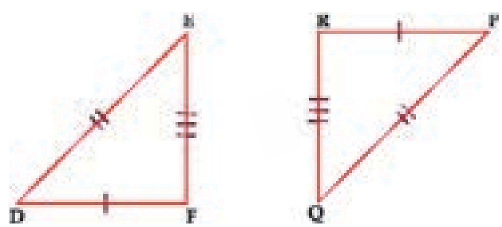


3. In the given figure, we have, $\Delta ACE \cong \Delta BDE$, such that $m\angle A = (3x + 1)°$, $m\angle B = (x + 35)°$, $m\angle AEC = (3y - 2)°$ and $m\angle DEB = (y + 8)°$.

   Find the values of x and y.

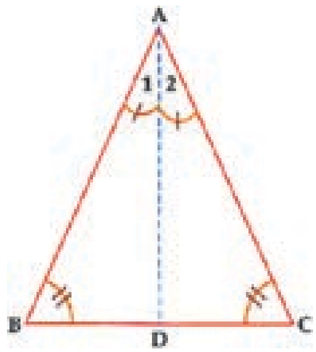


4. In the given figure,
   $\Delta DEF \cong \Delta PQR$, such that:
   $m\overline{DE} = (6x + 1)$cm, $m\overline{EF} = 8$cm,
   and $m\,\overline{RQ} = (5y - 7)$cm
   and $m\overline{PQ} = (10x - 19)$cm.
   Find the values of x and y.



Theorem 9.1.2

If two angles of a triangle are congruent, then the sides opposite to them are also congruent.



Given:

In ΔABC,

We have, $\angle B \cong \angle C$

To prove:

$\overline{AC} \cong \overline{AB}$

Construction: Draw $\overline{AD}$ the bisector of $\angle A$, meeting $\overline{BC}$ at point D.

Proof:

Statement	Reason
In $\Delta ADB \leftrightarrow \Delta ADC$	
i. $\angle B \cong \angle C$	i. Given
ii. $\angle 1 \cong \angle 2$	ii. Construction
iii. $\overline{AD} \cong \overline{AD}$	iii. Common side of both Δs (Identity congruence)
$\therefore$ $\Delta ADB \cong \Delta ADC$	A.S.A $\cong$ A.S.A
$\therefore$ $\overline{AB} \cong \overline{AC}$	Corresponding sides of congruent Δs

Q.E.D

Exercise 9.2

1. ABC is a triangle in which $m\angle A=35°$ and $m\angle B=100°$, $\overline{BD} \perp \overline{AC}$. Prove that ΔBDC is an isosceles triangle.

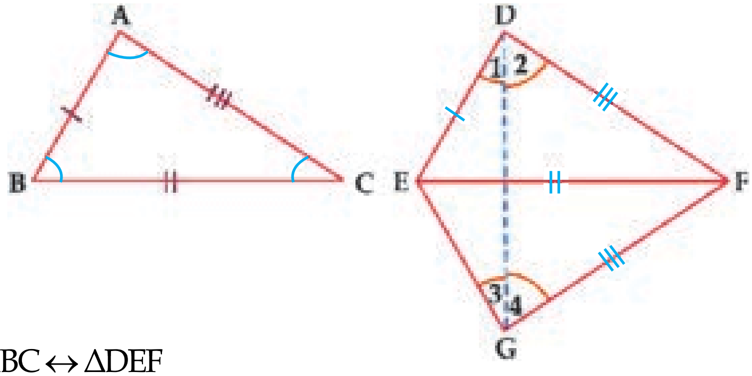
2. If the bisector of an angle of a triangle is perpendicular to its opposite side, then prove that triangle is an isosceles triangle.

3. ABC is a triangle in which $m\angle B = 45°$ and $\overline{CD} \perp \overline{AB}$. Prove that ΔDBC is an isosceles Δ.

Theorem 9.1.3

In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent.

Proof:



Given:

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$

To prove that: $\Delta ABC \cong \Delta DEF$

Construction: Suppose $\overline{BC}$ is the greatest of all the three sides of ΔABC. Construct ΔGEF such that:

i. Point G is on the opposite side of point D.
ii. $\angle FEG \cong \angle B$
iii. $\overline{EG} \cong \overline{BA}$
Join D and G.

Proof:

Statements	Reasons
1. In $\Delta ABC \leftrightarrow \Delta GEF$	**1.** Correspondence of two Δs
i. $\overline{BC} \cong \overline{EF}$	i. Given
ii. $\angle B \cong \angle GEF$	ii. Construction
iii. $\overline{BA} \cong \overline{GE}$	iii. Construction
2. $\therefore$ $\Delta ABC \cong \Delta GEF$	**2.** S.A.S. postulate.
3. $\therefore$ $\overline{AC} \cong \overline{GF}$ and $\angle A \cong \angle G$	**3.** By the congruence of triangles.
4. But $\overline{DF} \cong \overline{AC}$	**4.** Given
5. $\therefore$ $\overline{GF} \cong \overline{DF}$	**5.** Transitive property.
6. $\therefore$ In ΔDEG, $m\angle 1 = m\angle 3$	**6.** Opposite sides congruent $\overline{EG} \cong \overline{BA} \cong \overline{ED}$

7.	Similarly, in ΔGFD, $m\angle 2 = m\angle 4$	7.	$\overline{DF} \cong \overline{GF}$
8.	$\therefore\ m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	8.	Addition property of equation
9.	or $m\angle D = m\angle G$	9.	$m\angle 1 + m\angle 2 = m\angle D$ $m\angle 3 + m\angle 4 = m\angle G$
10.	But $m\angle G = m\angle A$	10.	Proved in (3) above
11.	$\therefore\ m\angle A = m\angle D$	11.	Transitive property in (3)
12.	In $\Delta ABC \leftrightarrow \Delta DEF$ i. $\overline{AB} \cong \overline{DE}$ ii. $\angle A \cong \angle D$ iii. $\overline{AC} \cong \overline{DF}$	12.	Correspondence of two Δs i. Given ii. Proved above iii. Given
13.	$\therefore\ \Delta ABC \cong \Delta DEF$	13.	S.A.S Postulate

Q.E.D

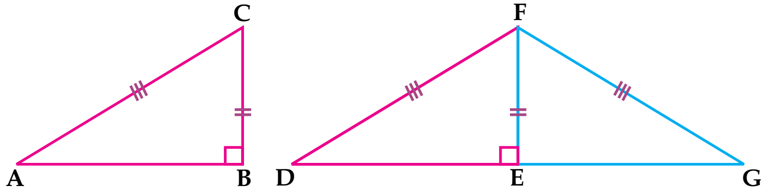
Corollary: The angles of an equilateral triangle are also equal in measurement.

Exercise 9.3

1. ABC is an isosceles triangle. D is the mid-point of base $\overline{BC}$. Prove that $\overline{AD}$ bisects $\angle A$ and $\overline{AD} \perp \overline{BC}$.
2. ABC and DBC are two isosceles triangles on the same side of a common base $\overline{BC}$. Prove that $\overline{AD}$ is the right bisector of $\overline{BC}$.
3. PQRS is a square. X,Y and Z are the mid-points of $\overline{PQ}$, $\overline{QR}$ and $\overline{RS}$ respectively. Prove that $\Delta PXY \cong \Delta SZY$.
4. Prove that, in an equilateral triangle any two median are congruent.

Theorem 9.1.4

If in the correspondence of two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent (H.S≅H.S).



Given: In correspondence
$\Delta ABC \leftrightarrow \Delta DEF$
$\angle B \cong \angle E$ (rt $\angle$s) $\overline{AC} \cong \overline{DF}$ (Hyp) and $\overline{BC} \cong \overline{EF}$

To Prove: $\Delta ABC \cong \Delta DEF$

Construction: Produce $\overline{DE}$ to point G such that $\overline{EG} \cong \overline{AB}$. Then join F and G.

Proof:

Statements	Reasons
$m\angle DEF + m\angle GEF = 180^\circ$	Supplement postulate
But $m\angle DEF = 90^\circ$	Given
$\therefore$ $m\angle GEF = 90^\circ$	$\therefore$ $180^\circ - 90^\circ = 90^\circ$
In $\Delta GEF \leftrightarrow \Delta ABC$	
i. $\overline{GE} \cong \overline{AB}$	Construction
ii. $\angle GEF \cong \angle ABC$	Each is right angle
iii. $\overline{EF} \cong \overline{BC}$	Given
$\therefore$ $\Delta GEF \cong \Delta ABC$	S.A.S. ≅ S.A.S.
$\therefore$ $\overline{FG} \cong \overline{AC}$ and $\angle G \cong \angle A$	By the congruence of Δs.
$\therefore$ $\overline{AC} \cong \overline{DF}$	$\therefore$ $\overline{AC} \cong \overline{DF}$ (Given)
In ΔDFG, $\angle D \cong \angle G$	Opposite sides congruents
$\therefore$ $\angle D \cong \angle A$	Each is congruent to ΔG
In $\Delta ABC \leftrightarrow \Delta DEF$	
i. $\angle A \cong \angle D$	i. Proved
ii. $\angle ABC \cong \angle DEF$	ii. rt Δs
iii. $\overline{AC} \cong \overline{DF}$	iii. Given
$\therefore$ $\Delta ABC \cong \Delta DEF$	A.A.S ≅ A.A.S

Q.E.D

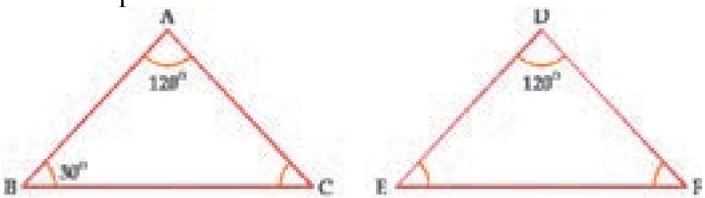
Note: Theorem 9.1.4 can be proved by S.A.S postulate.

Exercise 9.4

1. Prove that:
   The perpendiculars from the vertices of the base to opposite sides of an isoceles triangle are congruent. (Hint: Medians and altitudes of the triangle are congruent)
2. Prove that, if the bisector of an angle of a triangle bisects its opposite side, then the triangle will be an isosceles triangle.
3. Prove that the median bisecting the base of an isosceles triangle bisects the vertical angle and is perpendicular to the base.
4. Prove that if three altitudes of a triangle are congruent, then the triangle is equilateral.

Review Exercise 9

1. If $\Delta ABC \cong \Delta DEF$, $m\angle F$ is equal to
   A. 90°
   B. 60°
   C. 30°
   D. 20°



2. Identify true and false statement in the following:
   (i) The sum of the measure of all angles in an quadrilateral is 360°.
   (ii) The sum of the measure of all angles in a triangle is 270°.
   (iii) In an equilateral triangle, angles are of the same measurement.
   (iv) There are two right angles in a traiangle.
   (v) In an isosceles triangles, corresponding angles and correspoinding sides are equal in measure.
3. Fill in the blanks to make the sentences true sentences:
   (i) In $\Delta ABC \leftrightarrow \Delta DEF$, then $\overline{AC}$ corresponds to _____________.
   (ii) In $\Delta KLM \leftrightarrow \Delta PQR$, then $\angle MKL$ corresponds to _____________
   (iii) In an isosceles triangle, the base angle are _____________.
   (iv) If the mesure of each of the angles of a triangle is 60°, then the triangle is_____________.
   (v) In a right-angled triangle, side opposite to right angle is called _____________.
   (vi) The sum of the measures of acute angle of a right triangle is_____________.

4. Encircle the corresponding letters a,b,c or d for correct answer:

(i) Which of the following is not a sufficent condition for congurence of two triangles?

(a) A.S.A ≅A.S.A (b) H.S ≅ H.S
(c) S.A.A≅ S. A. A (d) A.A.A≅ A. A. A

(ii) In ΔABC, if ∠A ≅ ∠B, then the bisector of _____ angle divides the triangle into congruent triangles:

(a) ∠A (b) ∠B
(c) ∠C (d) any one of its angles.

(iii) The diagonal of ____ does not divide it into two congruent triangles:

(a) Rectangle (b) Trapezium
(c) Parallogram (d) Square

(iv) How many acute angles are there in an acute angled triangle?

(a) 1 (b) 2
(c) 3 (d) not more than 2.

Summary

In this unit we stated and proved the following theorem:

- In any correspondence of two traingles, if one side and any two angles of one traingle are congruent to the corresponding side and angles of the other, the two taringles are congruent. (A.S.A. ≅ A.S.A)
- If two angles of a traingles are congruent, then the side opposite to them are also congruent.
- In the correspondence of two traingles, if three sides of two traingles are congruent to the corresponding three sides of other, then the two traingles are congruent (S.S.S ≅ S.S.S).
- If in the correspondence of the two right-angled traingles the hypotenuse and one side of one traingles are congruent to the hypotenuse and the corresponding side of other, then the traingles are congruent. (H.S ≅ H.S).
- Two traingles are said to be congruent, if there exists a correspondence between them such that all the corresponding sides and traingles are congruent.(S.S.S).

Unit 10

• Weightage = 7%

PARALLELOGRAM AND TRIANGLES

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.

  a) In a parallelogram:
     - The opposite sides are congruent,
     - The opposites angles are congruent,
     - The diagonals bisect each other.

  b) If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.

  c) The line segments joining the midpoints of two sides of a triangle, is parallel to the third side and it is equal to one half of its length.

  d) The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

  e) If three or more parallel lines make congruent intercepts on the transversal, they also intercept congruent segments on any other line that cuts them.

Introduction

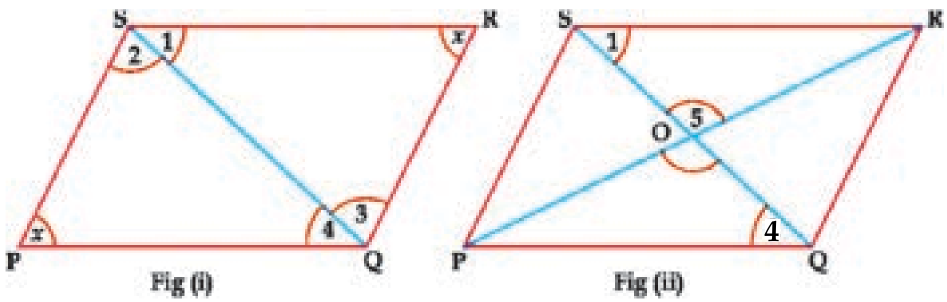
In the previous classes students learned and constructed many kinds of polygons like triangles, parallelogram, square, rectangle, rhombus, trapezium etc. Also observe the congruency related to their sides and angles. In this unit, we will discuss and understand the theorems related to parallelograms and Triangles.

10.1 Parallelograms and Triangles

Theorem 10.1.1

In a parallelogram:

- **The opposite sides are congruent,**
- **The opposites angles are congruent,**
- **The diagonals bisect each other.**



Given:

$||^{m}$ PQRS

To Prove:

1. $\overline{PQ} \cong \overline{RS}$; $\overline{PS} \cong \overline{QR}$
2. $\angle P \cong \angle R$; $\angle S \cong \angle Q$
3. Diagonals $\overline{PR}$ and $\overline{SQ}$ bisect each other at point O. [fig (ii)]

Construction:

In figure (i) join points S and Q.

Proof:

Statements	Reasons
In figure (i)	
(1)	
$\overline{SR} \parallel \overline{PQ}$, $\overline{SQ}$ is transversal,	Alternate angles of $\parallel$ lines
$m\angle 1 = m\angle 4$	
Similarly, $m\angle 2 = m\angle 3$	
$\therefore$ $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	Angle addition postulate
or $\angle S \cong \angle Q$	
Similarly, $\angle P \cong \angle R$	
i.e opposite angles are congruent	By above the same process
(2)	
$\Delta SPQ \leftrightarrow \Delta QRS$	
$\angle 1 \cong \angle 4$ and $\angle 2 \cong \angle 3$	Proved in (1) above
$\overline{SQ} \cong \overline{SQ}$	Common
$\therefore$ $\Delta SPQ \cong \Delta QRS$	A.S.A $\cong$ A.S.A
$\therefore$ $\overline{PQ} \cong \overline{RS}$ and $\overline{PS} \cong \overline{QR}$	By the congruence of Δs
i.e opposite sides are congruent	
In figure (ii)	
(3)	
$\Delta SOQ \leftrightarrow \Delta ROS$	
$\angle 1 \cong \angle 4$	Proved in (1) above
$\angle POQ \cong \angle SOR$	Vertically opposite angles
$\overline{PQ} \cong \overline{SR}$	Proved in (2) above
$\therefore$ $\Delta POQ \cong \Delta ROS$	A.A.S $\cong$ A.A.S
$\therefore$ $\overline{PO} \cong \overline{OR}$ and $\overline{OQ} \cong \overline{OS}$	By the congruence of Δs
$\therefore$ $\overline{PR}$ and $\overline{RS}$ diagonals bisect each other	

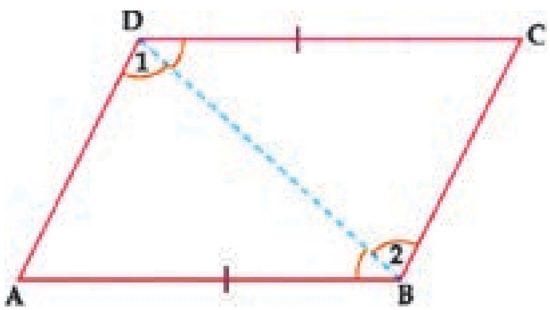
Q.E.D

Exercise 10.1

1. The line joining the mid-points of two opposite sides of parallelogram is parallel to the other sides.
2. Interior angles on any side of a parallelogram are supplementary.
3. Prove that the bisectors of two angles on same side of a parallelogram cut each other at right angle.
4. If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
5. In parallelogram opposite angles are congruent.

Theorem 10.1.2

If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.



Given:

In a quadrilateral ABCD,

$\overline{AB} \cong \overline{CD}$ and $\overline{AB} \parallel \overline{CD}$

To prove:

Quadrilateral ABCD is a $||^m$

Construction: Join B and D.

Proof:

Statements	Reasons
1. $\overline{AB} \parallel \overline{CD}$, $\overline{BD}$ is transversal $\therefore$ $\angle ABD \cong \angle CDB$	1. Alternate angles of $\parallel$ lines
2. In $\Delta ADB \leftrightarrow \Delta CBD$	2. Correspondence of two Δs.
i. $\overline{AB} \cong \overline{CD}$	i. Given
ii. $\angle ABD \cong \angle CDB$	ii. Proved above
iii. $\overline{BD} \cong \overline{BD}$	iii. Common
3. $\therefore$ $\Delta ADB \cong \Delta CBD$	3. S.A.S $\cong$ S.A.S
4. $\therefore$ $\angle 1 \cong \angle 2$	4. By the congruence of triangles.
5. But these are alternate angles	5. By definition of alternate angles.
6. $\overline{AD} \parallel \overline{BC}$	6. Alternate angles are congruent
7. $\overline{AB} \parallel \overline{CD}$	7. Given
8. ABCD is a $||^m$	8. Opposite sides are parallel

Q.E.D

Exercise 10.2

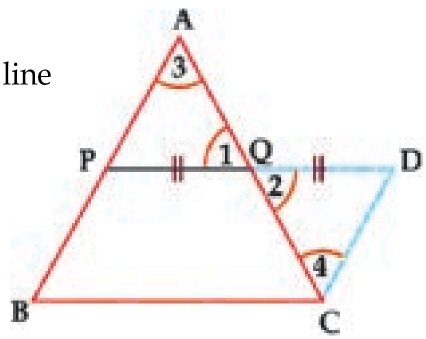
1. Prove that a quadrilateral is a parallelogram, if its opposite angles are congruent
2. Prove that a quadrilateral is a parallelogram, if its diagonals bisect each other.
3. If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.
4. If a quadrilateral is a parallelogram, it has diagonals which form two congruent triangles.
5. If the angles formed with every side of a quadrilateral are supplementary, it is a parallelogram.

Theorem 10.1.3

The line segment joining the mid-points of two sides of a triangle, is parallel to the third side and it is equal to one half of its length.

Given:

P and Q are midpoints of $\overline{AB}$ and $\overline{AC}$ in ΔABC, respectively and $\overline{PQ}$ is the line segment joining them.



To prove:

$\overline{PQ} \parallel \overline{BC}$ and $m\overline{PQ} = \frac{1}{2} m\overline{BC}$

Construction:

Produce $\overrightarrow{PQ}$ to D such that $\overline{QD} \cong \overline{PQ}$.
Join D and C.

Proof:

Statements	Reasons
In $\Delta APQ \leftrightarrow \Delta CDQ$	
i. $\overline{PQ} \cong \overline{QD}$	i. Construction
ii. $\angle 1 \cong \angle 2$	ii. Vertical angles
iii. $\overline{AQ} \cong \overline{QC}$	iii. Given

$\therefore$ $\Delta APQ \cong \Delta CDQ$	S.A.S postulate
$\therefore$ $\overline{AP} \cong \overline{CD}$ and $\angle 3 \cong \angle 4$	By the congruence of Δs
But $\overline{PB} \cong \overline{AP}$	Given
$\therefore$ $\overline{PB} \cong \overline{CD}$	Each is congruent to $\overline{AP}$
$\angle 3$ and $\angle 4$ are alternate $\angle$s	By definition of alternate $\angle$s
$\therefore$ $\overline{AB} \parallel \overline{CD}$ i.e. $\overline{PB} \parallel \overline{CD}$	Alternate $\angle$s are congruent
$\therefore$ PBCD is a $\parallel^{m}$	A pair of opposite sides $\parallel$ and $\cong$
$\therefore$ $\overline{PD} \parallel \overline{BC}$ and $\overline{PD} \cong \overline{BC}$	Opposite sides of $\parallel^{m}$ are parallel and congruent.
$\therefore$ $\overline{PQ} \parallel \overline{BC}$	$\overleftrightarrow{PD}$ and $\overleftrightarrow{PQ}$ are same line and
and $m\overline{PQ} = \frac{1}{2} m\overline{BC}$	$m\overline{PQ} = m\overline{QD} = \frac{1}{2} m\overline{PD}$

Q.E.D

Exercise 10.3

1. If the line segments joining the mid-points of two sides of a triangle, is parallel to the third side and its length is 4 cm. What is the length of third side.
2. Prove that the line-segment joining the mid-points of the opposite sides of a quadrilateral bisect each other.
3. The line segments, joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram.
4. Prove that the line segment passing through the mid-point of one side and parallel to another side of a triangle also bisect the third side.
5. Prove that four triangles obtained by joining the mid-points of the three sides of a triangle are all congruent to each other.

Theorem 10.1.4

The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.

Given:

ΔABC, in which medians $\overline{BE}$ and $\overline{CF}$ meet in G.

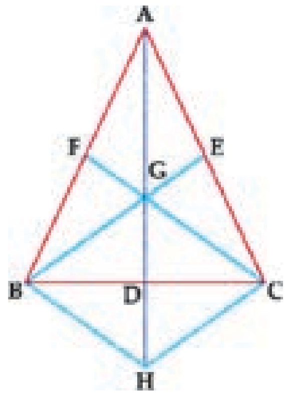
To prove:

i. $\overline{AG}$ bisects $\overline{BC}$ in D, and

ii. G is the point of trisection of each median.

Construction:

Draw $\overline{CH} \parallel \overline{EB}$ meeting $\overline{AD}$ produced in H. Join points B and H.



Proof:

Statements	Reasons
In ΔACH, $\overline{AE} \cong \overline{EC}$	Given
and $\overline{EG} \parallel \overline{CH}$	Construction
$\therefore$ $\overline{AG} \cong \overline{GH}$	By converse of theorem 10.1.3
Further in ΔABH	
$\overline{AG} \cong \overline{GH}$	Proved above
$\overline{AF} \cong \overline{FB}$	Given
$\overline{FG} \parallel \overline{BH}$	By theorem 10.1.3
Hence BGCH is a $\parallel^m$	Opposite side are parallel
Diagonals $\overline{BC}$ and $\overline{GH}$ bisect each other	By theorem 10.1.1
i.e. $\overline{GD} \cong \overline{DH}$, $\overline{BD} \cong \overline{DC}$	
$\overline{AD}$ median of ΔABC	$\overline{BD} \cong \overline{DC}$ (Proved above)
Also $m\overline{AG} = m\overline{GH} = 2m\overline{GD}$	$\overline{GD} \cong \overline{DH} \Rightarrow m\overline{GH} = 2m\overline{GD}$
G is a point of trisection of $\overline{AD}$	As $\overline{AG}$ is twice of $\overline{GD}$
Similarly we can prove that G is a point of trisection of $\overline{BE}$ and $\overline{CF}$ as well	By the above process

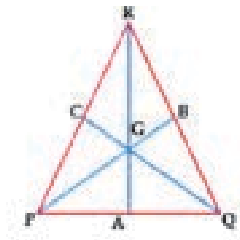
Q.E.D

Exercise 10.4

1. If three medians of a triangle are congruent, prove that the triangle is an equilateral.

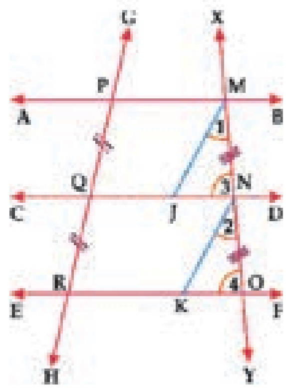
2. The medians $\overline{PB}, \overline{QC}$ and $\overline{RA}$ of ΔPQR meet in point G, show that G is the centroid of ΔPQR.

3. In the given figure, the length $\overline{GR}$ is of 2cm then find the length of $\overline{AG}$.



Theorem 10.1.5

If three or more parallel lines make congruent intercepts on the transversal, they also intercept congruent segments on any other line that cuts them.



Given:

$\overleftrightarrow{AB}, \overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ cut transversal $\overleftrightarrow{GH}$ at points P, Q and R respectively such that:

$\overline{PQ} \cong \overline{QR}$

$\overleftrightarrow{XY}$ is another transversal cutting $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ in points M, N, O respectively.

To prove: $\overline{NM} \cong \overline{NO}$

Construction:

Draw $\overline{MJ}$ and $\overline{NK}$ each parallel to $\overleftrightarrow{GH}$ meeting $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ respectively in points J and K.

Proof:

Statements	Reasons
$\overline{PM} \parallel \overline{QJ}$	$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ (Given)
and $\overline{PQ} \parallel \overline{MJ}$	Construction
$\therefore$ PMJQ is a $\parallel^m$	Opposite sides parallel
$\therefore$ $\overline{PQ} \cong \overline{MJ}$	Opposite sides of a $\parallel^m$
Similarly, QRKN is a $\parallel^m$	$\overline{QR} \parallel \overline{NK}$ and $\overline{QN} \parallel \overline{RK}$

Statements	Reasons
$\therefore$ $\overline{QR} \cong \overline{NK}$	By above reason
But $\overline{PQ} \cong \overline{QR}$	Given
$\therefore$ $\overline{MJ} \cong \overline{NK}$	Transitive property of equality
Now $\overline{MJ} \parallel \overline{NK}$	Each paralle to $\overline{GH}$
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding $\angle$s of $\parallel$lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$
In $\Delta MNJ \leftrightarrow \Delta NOK$	
i. $\angle 1 \cong \angle 2$	i. Proved above
ii. $\angle 3 \cong \angle 4$	ii. Corresponding $\angle$s of $\parallel$lines $\overleftrightarrow{MJ} \cong \overleftrightarrow{NK}$
iii. $\overline{MJ} \cong \overline{NK}$	iii. Proved above
$\therefore$ $\Delta MNJ \cong \Delta NOK$	A.A.S $\cong$ A.A.S
$\therefore$ $\overline{MN} \cong \overline{NO}$	By the congruence of Δs.

Q.E.D

Exercise 10.5

1. The triangle formed by joining the mid-points of the sides of a triangle is equivalent to the original triangle.
2. The line segment joining the mid-points of the non-parallel sides of a trapezium is parallel to the parallel sides and is equal to half their sum.
3. Every line segment drawn from the vertex to the base of a triangle is bisected by the line joining the mid-point of the other two sides.

Review Exercise 10

1. **Fill in the blanks:**
   (i) In a parallelogram, opposite sides are ______________.
   (ii) In a parallelogram, opposite angles are ______________.
   (iii) In a triangle, medians are ______________.
   (iv) In a parallelogram, the diagonals ______________ each other.
   (v) In a parallelogram, corresponding angles are ______________.
   (vi) Sum of the measures of interior angles of a quadrilateral is equal to __.

2. Tick (✓) the correct answer.

(i) Diagonals of a square are _______ to each other.
   a) Perpendicular   b) Non congruent
   c) Congruent   d) Both 'a' and 'c'

(ii) Sum of the measures of interior angles of a quadrilateral is
   a) 2 right angles   b) 4 right angles
   c) 3 right angles   d) none of these

(iii) Measure of a line segment joining the mid points of $\overline{AB}$ and $\overline{AC}$ of ΔABC is 3.5cm, then $m\overline{BC} =$
   a) 4.5cm   b) 5.5cm
   c) 6cm   d) 7cm

(iv) Two medians $\overline{AD}$ and $\overline{BE}$ of ΔABC intersect each other at G. If $m\overline{GD} = 1.7$cm, then $m\overline{AG} =$
   a) 2.7cm   b) 8.85cm
   c) 3.4cm   d) 5.1cm

(v) If sum of the measures of $\angle A$ and $\angle C$ of a parallelogram ABCD is 130°, then $m\angle B =$
   a) 25°   b) 115°   c) 65°   d) none of these

(vi) If opposite angles of a quadrilateral are equal in measures and none of them is a right angle, then the quadrilateral is a
   a) Square   b) Parallelogram
   c) Trapezium   d) Rectangle

(vii) Centroid is the common point of intersection of
   a) Medians of a triangle
   b) Diagonals of a parallelogram
   c) Angle bisectors of a triangle
   d) Perpendicular bisectors of a triangle

(viii) A point on median of a triangle is
   a) equidistant from its vertices
   b) equidistant from the mid points of its sides
   c) equidistant from its altitudes
   d) none of these

Summary

- Opposite sides of parallelogram are congruent.
- Opposite angles of parallelogram are congruent.
- Supplementary angles property holds for consecutive angles.
- Diagonals of a parallelogram bisect each other and each diagonal separates it into two congruent triangles
- If one angle of a parallelogram is right angle, then all the angles are right angles.
- Diagonals of a parallelogram divide it into four congruent triangles.
- Sum of the angles of a parallelogram is 360°.
- Sum of the interior angles of a triangle is 180°.
- Sum of the exterior angles of a triangle is 360°.
- If three or more parallel lines make congruent segments on a transversal they also intercept congruent segments on any other lines that cuts them.
- The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.
- The line segment joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half of its length.

Unit 11

• Weightage = 7%

LINE BISECTORS AND ANGLE BISECTORS

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point equidistant from the points of a line segment is on its right bisector.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on its bisector.
- The bisectors of the angles of a triangle are concurrent.

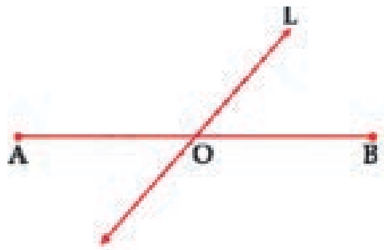
Introduction

We will discuss here the theorems and problems related to line bisector and angle bisector.

Definitions:

i) Bisector of a line segment.

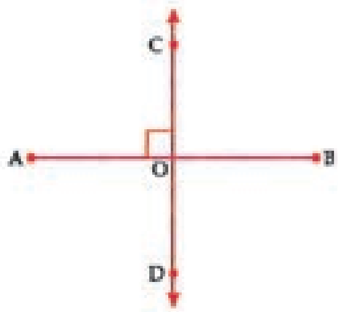
A line, ray or segment is called bisector which cuts another line segment into two equal parts.



For example, in the given figure, a bisector of a line segment AB is a line '**L**' that passes through the mid-point 'O' of the $\overline{AB}$.

ii) Right bisector of a line segment.

A right bisector of line segment can be defined as a line which divides a line segment into two equal parts at an angle of 90 degrees.

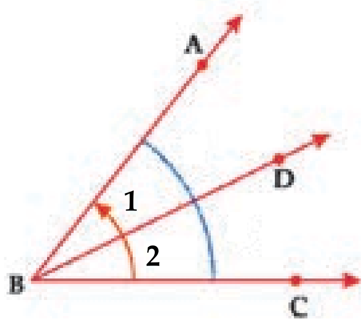


For example, in the given figure, $\overleftrightarrow{CD}$ is perpendicular to the line segment AB and it passes through its mid-point 'O'. Then $\overleftrightarrow{CD}$ is right bisector of $\overline{AB}$.

In the given figure, a line $\overleftrightarrow{CD}$ is a right bisector of $\overline{AB}$.

iii) Bisector of an angle.

A line or ray or line segment is called a **bisector of an angle** or **angle bisector,** if it divides the angle into two equal angles.

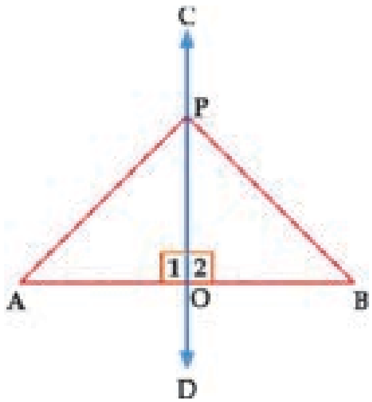


In the given figure, $\overrightarrow{BD}$ is an angle bisector of $\angle CBA$. $\overrightarrow{BD}$ divides $\angle CBA$ into two equal angles $\angle 1$ and $\angle 2$ i.e. $\angle 1 \cong \angle 2$.

Theorem 11.1.1

Prove that:

Any point on the right bisector of a line segment is equidistant from its end points.



Given:

$\overleftrightarrow{CD}$ is the right bisector of $\overline{AB}$ intersecting it at O. P is any point on $\overleftrightarrow{CD}$.

To Prove: $\overline{AP} \cong \overline{BP}$, i.e. P is equidistant from A and B.

Proof:

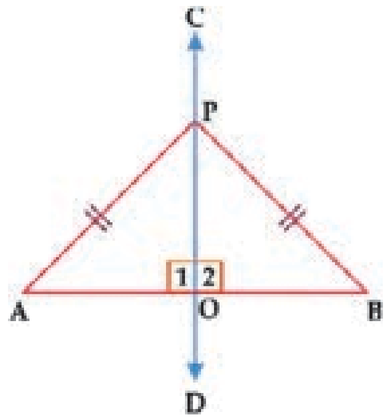
Statements	Reasons
1. In $\Delta AOP \leftrightarrow \Delta BOP$ i. $\overline{AO} \cong \overline{OB}$ ii. $\angle 1 \cong \angle 2$ iii. $\overline{PO} \cong \overline{PO}$	**1.** i. Given (O is the mid-point) ii. Given ($\overline{CD} \perp \overline{AB}$ at O) iii. Common
2. $\therefore$ $\Delta AOP \cong \Delta BOP$	**2.** S.A.S postulate
3. $\therefore$ $\overline{AP} \cong \overline{BP}$	**3.** Corresponding sides of congruent Δs.
4. But P is an arbitrary point on $\overleftrightarrow{CD}$ Similarly any other point on CD is equidistant A and B. Hence, every point on the right bisector is equidistant from its end points.	**4.** By assumption By the above process.

Q.E.D

Theorem 11.1.2

Prove that:

Any point equidistant from end points of a line segment is on the right bisector of it. (Converse of the theorem 11.1)



Given:

A and B are two fixed points and P is a moving point such that $\overline{PA} \cong \overline{PB}$

To Prove:

P lies on the right bisector of $\overline{AB}$.

Construction:

Bisect $\overline{AB}$ at O. Join points P and O.

Proof:

Statements	Reasons
1. In $\Delta POA \leftrightarrow \Delta POB$	1. Correspondence of two Δs.
i. $\overline{AO} \cong \overline{OB}$	i. Construction
ii. $\overline{PA} \cong \overline{PB}$	ii. Given
iii. $\overline{PO} \cong \overline{PO}$	iii. Common side of both Δs
2. $\Delta POA \cong \Delta BOP$	2. S.S.S $\cong$ S.S.S
3. $\angle 1 \cong \angle 2$	3. Corresponding $\angle$s. of congruent Δs.
4. But $\angle 1$ and $\angle 2$ are supplementary $\angle$s.	4. $\overline{AB}$ is a line (supplement postulate)
5. Each $\angle 1$ and $\angle 2$ is right angle.	5. If two supplementary angles are equal in measure each is right angle.
6. Thus $\overline{PO}$ is the right bisector of $\overline{AB}$.	6. $\overline{PO} \perp \overline{AB}$ and $\overline{AO} \cong \overline{BO}$
7. Thus every point equidistant from points A and B is on the right bisector of $\overline{AB}$.	7. We can prove by the above process.

Q.E.D

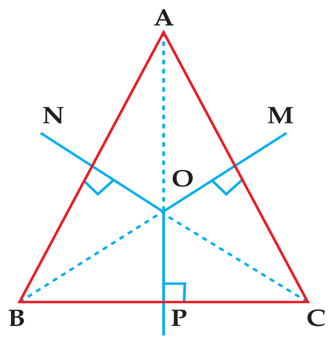
Exercise 11.1

1. Prove that the point of intersection of the right bisector of any two sides of a triangle is equidistance from all the vertices of the triangle.
2. Prove that the centre of the circle is on the right bisectors of each of its chords.
3. Where will be the centre of a circle passing through three non-collinear points and why?
4. If two circles intersect each other at points A and B then prove that the line passing through their centres will be the right bisector of $\overline{AB}$.
5. Three markets A, B and C are not on the same line. The business men of these markets want to construct a Masjid at such a place which is equidistant from these markets. After deciding the place of Masjid, prove that this place is equidistant from all the three markets.

Theorem 11.1.3

Prove that:

The right bisectors of the sides of a triangle are concurrent.



Given:

A triangle ABC

To Prove:

The right bisectors of the sides of a triangle are concurrent.

Construction:

Draw $\overline{NO}$, $\overline{MO}$, the right bisectors of $\overline{AB}$ and $\overline{AC}$ meeting in O. Bisect $\overline{BC}$ at P. Draw $\overline{OP}$, $\overline{OA}$, $\overline{OB}$, $\overline{OC}$.

Proof:

Statements	Reasons
1. $\overline{NO}$ is right bisector of $\overline{AB}$	1. Construction
2. ∴ $\overline{AO} \cong \overline{OB}$	2. By theorem 11.1.1
3. Similarly, $\overline{AO} \cong \overline{OC}$	3. $\overline{MO}$ is right bisector of $\overline{AC}$.
4. ∴ $\overline{OB} \cong \overline{OC}$	4. Each is congruent to $\overline{AO}$.
5. P is the mid-point of $\overline{BC}$.	5. Construction
6. ∴ $\overline{OP}$ is the right bisector of $\overline{BC}$	6. By theorem 11.1.2
7. Hence right bisector of the sides of a triangle are concurrent.	7. All of them meet in one point.

Q.E.D

Exercise 11.2

1. Prove that in an acute triangle the circumcenter falls in the interior of the triangle.
2. Prove that the right bisectors of the four sides of an isosceles trapezium are concurrent.
3. Prove that the altitudes of a triangle are concurrent.

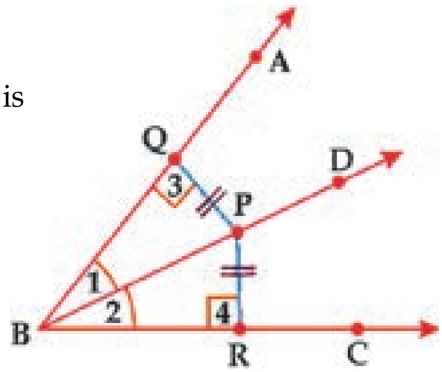
Theorem 11.1.4

Prove that:

Any point on the bisector of an angle is equidistant from its arms.

Given:

$\overrightarrow{BD}$ is the angle bisector of $\angle ABC$. P is any point on $\overrightarrow{BD}$. $\overline{PQ}$ and $\overline{PR}$ are perpendiculars on $\overrightarrow{BA}$ and $\overrightarrow{BC}$ respectively.



To prove:

$\overline{PQ} \cong \overline{PR}$ (i.e. point P is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$)

Proof:

Statements	Reasons
1. In $\Delta PQB \leftrightarrow \Delta PRB$	**1.**
i. $\angle 3 \cong \angle 4$	i. Each is a right angle
ii. $\angle 1 \cong \angle 2$	ii. $\overrightarrow{BD}$ is the angle bisector (Given)
iii. $\overline{BP} \cong \overline{BP}$	iii. Common side of both Δs.
2. $\Delta PQB \cong \Delta PRB$	**2.** A.A.S≅ A.A.S
3. $\overline{PQ} \cong \overline{PR}$ (i.e. P is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$)	**3.** Corresponding sides of congurent Δs.

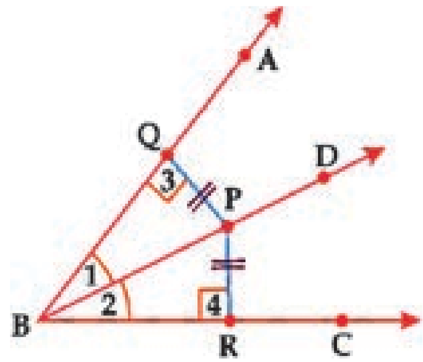
Q.E.D

Theorem 11.1.5

Prove that:

Any point inside an angle, equidistant from its arms, is on the bisector of it. (Converse of Theorem 11.4)

Given:



P is any point of $\overrightarrow{BD}$ equidistant from the arms $\overrightarrow{BA}$ and $\overrightarrow{BC}$ of $\angle ABC$, i.e. $\overline{PQ} \cong \overline{PR}$ and $\overline{PQ} \perp \overrightarrow{BA}$ and $\overline{PR} \perp \overrightarrow{BC}$.

To Prove:

$\angle 1 \cong \angle 2$, i.e. $\overrightarrow{BD}$ is the bisector of $\angle ABC$.

Proof:

Statements	Reasons
1. In $\Delta PQB \leftrightarrow \Delta PRB$	**1.** Correspondence in right Δs
i. $\angle 3 \cong \angle 4$	i. Each is a right angle
ii. $\overline{PQ} \cong \overline{PR}$	ii. Given
iii. $\overline{BP} \cong \overline{BP}$	iii. Common hypotenuse
2. $\Delta PQB \cong \Delta PRB$	**2.** In rt. Δs H.S $\cong$ H.S
3. $\angle 1 \cong \angle 2$, (i.e. $\overrightarrow{BD}$ is the bisector of $\angle ABC$)	**3.** Corresponding $\angle$s of congruent Δs.

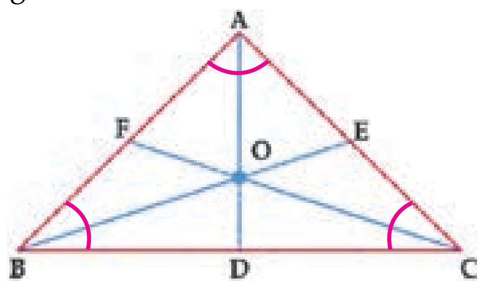
Q.E.D

Theorem 11.1.6

Prove that:

The bisectors of the angles of a triangle are concurrent.

Given:



In ΔABC, $\overline{BE}$ and $\overline{CF}$ are the bisectors of $\angle B$ and $\angle C$ respectively which intersect each other at point 'O'.

To Prove:

The bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Construction:

Draw $\overline{OF} \perp \overline{AB}$ and $\overline{OD} \perp \overline{BC}$.

Proof:

Statements	Reasons
In correspondence $\overline{OD} \cong \overline{OF}$ ... (i)	A point on bisector of an angle is equidistant from its arm.
Similarly $\overline{OD} \cong \overline{OE}$ ... (ii)	
$\therefore$ $\overline{OE} \cong \overline{OF}$	From (i) and (ii)
So, the point O is on the bisector of $\angle A$.	Theorem 11.1.5
Also the point O is on the bisectors of $\angle ABC$ and $\angle BCA$	Given
Thus, the bisectors of $\angle A, \angle B$ and $\angle C$ are concurrent at O.	

Q.E.D

Exercise 11.3

1. Two isosceles triangles have a common base, prove that the line joining vertices bisects the common base at right angle.
2. If the bisector of an angle of a triangle bisects the opposite side, prove that triangle is an isosceles.
3. In an isosceles ΔABC, $m\overline{AB} = m\overline{AC}$. Prove that the perpendiculars from the vertices B and C to their opposite sides are equal.

Review Exercise 11

1. Prove that, if two altitudes of a triangle are congruent, the triangle is an isosceles.
2. Prove that, a point in the interior of a triangle is an equidistant from all the three sides' lies on the bisector of all the three angles of the triangle.
3. Write **'T'** for True and **'F'** for False in front of each of the following statements
   (i) Bisection of side means, we divide the given side into two equal parts.
   (ii) In a right angled isosceles triangle each angle on the base is of 45° .
   (iii) Triangle of congruent sides has congruent angles.
4. Choose the correct option:

(i) There are __________ acute angles in an acute angled triangle.
(a) One (b) Two (c) Three (d) None

(ii) An point equidistant from the end points of a line segment is on the __________ of it.
(a) Right bisector (b) Perpendicular
(c) Centre (d) Mid-point

(iii) __________ of the sides of an acute angled triangle intersect each other inside the triangle
(a) Perpendicular (b) The right bisector
(c) Obtuse (d) Acute

(v)The bisector of the angles of a triangle are __________.
(a) Concurrent (b) Collinear
(c) Do not interest (d) Unequal

Summary

- A bisector of a line segment divides the line segment into two equal parts
- Right bisector cuts the line segment into two equal parts at 90°.
- Any point on the right bisector of a line segment is equidistant from its end points.
- Any point is equidistant from the points of a line segment is on its right bisector.
- The right bisectors of the sides of a triangle are concurrent.
- Any point on the bisector of an angle is equidistant from its arms.
- Any point inside an angle, equidistant from its arms, is on its bisector.
- The bisectors of the angles of a triangle are concurrent.

Unit 12

• Weightage = 5%

SIDES AND ANGLES OF A TRAINGLE

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Introduction

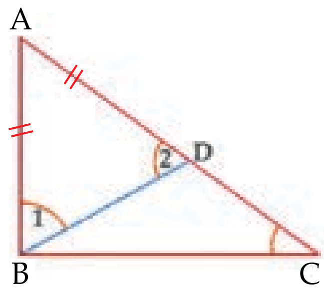
In this unit we will learn the theorems related to the sides and angles of the triangle along with their corollaries and apply them to solve the allied problems.

12.1 Sides and Angles of a Triangle

Theorem 12.1.1

Prove that:

If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.



Given:

In ΔABC, $m\overline{AC} > m\,\overline{AB}$.

To Prove:

$$m\angle ABC > m\angle ACB$$

Construction:

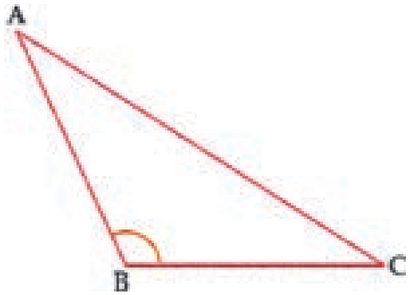
On $\overline{AC}$ take a point D such that $\overline{AD} \cong \overline{AB}$.
Join B to D so that ΔADB is an isosceles triangle.
Label $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements	Reasons
In ΔABD, $m\angle 1 = m\angle 2$ ... (i)	Angles opposite to congruent sides (construction).
In ΔBCD, $m\angle ACB < m\angle 2$ or $m\angle 2 > m\angle ACB$ ... (ii)	(An exterior angle of a triangle is greater than a non-adjacent interior angles)
$\therefore$ $m\angle 1 > m\angle ACB$ ... (iii)	By (i) and (ii)
But $\angle ABC = m\angle 1 + m\angle DBC$ $\therefore$ $m\angle ABC > m\angle 1$ ... (iv)	Postulate of addition of angles
$\therefore$ $m\angle ABC > m\angle 1 > m\angle ACB$	By (iii) and (iv)
Hence $\angle ABC > m\angle ACB$	(Transitive property of in-equality of real numbers).

Following example will help to understand the above theorem.

Example: Prove that in a scalene triangle, the angle opposite to the largest side is of measure greater than 60°.



Given:

In ΔABC, with, $m\overline{AC} > m\overline{AB}$ and $m\overline{AC} > m\overline{BC}$.

To Prove:

$m\angle B > 60^0$

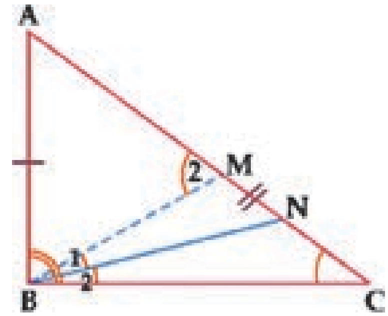
Proof:

Statements	Reasons
In ΔABC.	
We have, $m\angle B > m\angle C$	$m\overline{AC} > m\overline{AB}$
and $m\angle B > m\angle A$	$m\overline{AC} > m\overline{BC}$ } Given
but, $m\angle A + m\angle B + m\angle C = 180^0$	$\angle A$, $\angle B$ and $\angle C$ are the angles of the ΔABC.
$\therefore$ $m\angle B + m\angle B + m\angle B > 180^0$	$m\angle B > m\angle C$ and $m\angle B > m\angle A$
i.e $3\,m\angle B > 180^0$	By addition
$=$ $m\angle B > \frac{180^0}{3}$	Dividing both sides by 3
Thus, $m\angle B > 60^0$	

Q.E.D

Theorem 12.1.2

Prove that if two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.



Given:

In ΔABC, $m\angle B > m\angle C$

To Prove:

$m\overline{AC} > m\overline{AB}$

Construction:

Make $\angle ABM \cong \angle C$. Draw $\overline{BN}$, the bisector of $\angle MBC$, i.e. $m\angle 1 = m\angle 2$.

Proof:

Statements	Reasons
$\angle ANB$ is the exterior $\angle$ of ΔCBN	By definition of exterior $\angle$
$\therefore\ m\angle ANB = m\angle C + m\angle 2$	
$= m\angle C + m\angle 1$	$\therefore\ m\angle 2 = m\angle 1$ (Construction)
$= m\angle ABM + m\angle 1$	$\therefore\ m\angle C = m\angle ABM$ (Construction)
$= m\angle ABN$	By angle addition postulate
$\therefore\ \overline{AB} \cong \overline{AN}$	$\therefore\ m\angle ANB = m\angle ABN$ (Proved above)
$\therefore\ m\overline{AC} > m\overline{AB}$	$\therefore\ m\overline{AC} > m\overline{AN}$

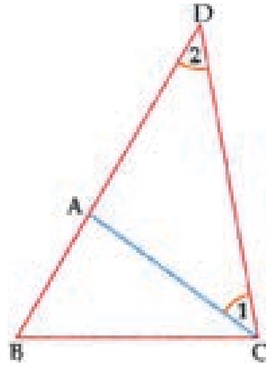
Q.E.D

Corollaries:

1. The hypotenuse of a right angle is longer than each of the other two sides.
2. In an obtuse angled triangle, the side opposite to the obtuse angle is longer than each of the other two sides.

Theorem 12.1.3

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



Given:

ΔABC

To Prove:

i) $m\overline{AB} + m\overline{AC} > m\overline{BC}$

ii) $m\overline{AB} + m\overline{BC} > m\overline{CA}$

iii) $m\overline{AC} + m\overline{BC} > m\overline{AB}$

Construction:

Produce $\overrightarrow{BA}$ to D, making $\overline{AD} \cong \overline{AC}$. Draw $\overline{DC}$

Proof:

Statements	Reasons
In ΔADC, $\overline{AD} \cong \overline{AC}$	Construction
$\therefore$ $m\angle 1 = m\angle 2$	Angles opposite to congruent sides
But $m\angle BCD > m\angle 1$	$m\angle BCD = m\angle BCA + m\angle 1$
$\therefore$ $m\angle BCD > m\angle 2$	Transitive property of inequality
$\therefore$ In ΔBDC, $m\overline{BD} > m\overline{BC}$ ...(i)	Grater angle has greater side opposite to it.
But $m\overline{BD} = m\overline{AB} + m\overline{AD}$	By construction
$= m\overline{AB} + m\overline{AC}$	$m\overline{AD} = m\overline{AC}$
$\therefore$ $m\overline{AB} + m\overline{AC} > m\overline{BC}$	Putting value of $\overline{BD}$ in (i)
Similarly, we can prove that:	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	By the above process
and $m\overline{BC} + m\overline{AC} > m\overline{AB}$	

Q.E.D

The following example will help to understand the above theorem.

Example 01 Which of the following sets of lengths of the sides form a triangle:

(i) 3 cm, 4 cm and 5 cm **(ii)** 4 cm, 5 cm and 4.5 cm

(iii) 60 mm, 80 mm and 10 cm **(iv)** 3 cm, 4 cm and 10 cm

Solution:

(i) 3 cm, 4 cm and 5 cm

Since, $3 + 4 > 5$, $3 + 5 > 4$ and $4 + 5 > 3$

$\therefore$ the sum of the two sides of greater than the 3rd side.

Thus, the given set of lengths form a triangle.

(ii) 4 cm, 5 cm and 4.5 cm

Since, $4 + 5 > 4.5$, $5 + 4.5 > 4$ and $4.5 + 3 > 4$

Thus, the given set of lengths form a triangle.

(iii) 60 mm, 80 mm and 10 cm

Since, 10 mm = 1 cm so, 60 mm = 6 cm and 80 mm = 8 cm

Now, $6 + 8 > 10$, $6 + 10 > 8$ and also $8 + 10 > 6$

Thus, the given set of lengths form a triangle.

Example 02 By using the idea of the above theorem decide, which of the following sets of lengths of the sides form a triangle:

(i) 2 cm, 4 cm and 7 cm **(ii)** 5.5 cm, 5 cm and 9.5 cm

Solution:

(i) 2 cm, 4 cm and 7 cm

Since, $2 + 4 < 7$, $4 + 7 > 2$ and $7 + 2 > 4$

Thus, this type of set of lengths cannot form a triangle.

(ii) 5.5 cm, 5 cm and 9.5 cm

Since, $5.5 + 5 > 9.5$, $5 + 9.5 > 5.5$ and $9.5 + 5.5 > 5$

Thus, the given set of lengths form a triangle.

Activity

If $a = 3$ cm

$b = 4$ cm

$c = 5$ cm

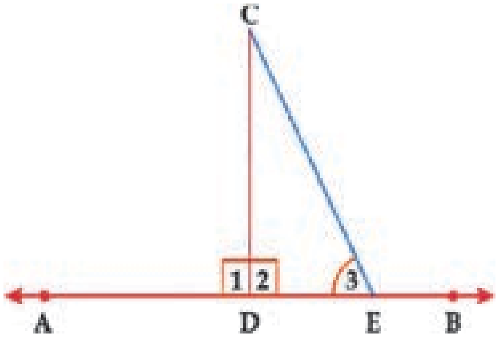
then ΔABC can be formed or not.

Theorem 12.1.4: Prove that:

From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Given:

From a point C, $\overline{CD}$ is drawn perpendicular to $\overleftrightarrow{AB}$ meeting it in D and $\overline{CE}$ is any other segment meeting $\overleftrightarrow{AB}$ in E.



To Prove:

$m\overline{CD} < m\overline{CE}$

Proof:

Statements	Reasons
$\angle 1$ is an exterior $\angle$ of ΔCDE	By definition of exterior $\angle$
$\therefore \quad m\angle 1 > m\angle 3$	Exterior $\angle$ is greater than non-adjacent interior $\angle$
$\therefore \quad m\angle 2 > m\angle 3$	$m\angle 1 = m\angle 2$(right $\angle$s)
$\therefore \quad m\overline{CE} > m\overline{CD}$	Side opposite to greater angle
Similarly, it can be proved that $m\overline{CD}$ is less than any other segment drawn from C to $\overleftrightarrow{AB}$	By the above process

Q.E.D

Corollaries:

1. The distance between a line and point (on a line) is zero.

Exercise 12.1

1. **O** is an interior point of the ΔABC.

   Show that: $m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$.

2. In ΔABC, $m\angle B = 70^{\circ}$ and $m\angle C = 45^{\circ}$. Which of the sides of the triangle is longest?

3. In ΔABC, $m\angle A = 55^{\circ}$ and $m\angle B = 65^{\circ}$, which of the side of the triangle is smallest?

Review Exercise 12

1. **Tick (✓) True or False from the following statements.**
   - (i) Sum of the two sides of a triangle is greater than the third side. **T /F**
   - (ii) The difference of two sides of a triangle is larger than the third side. **T /F**
   - (iii) Perpendicular distance from a point to line is the longest distance between them. **T /F**
   - (iv) In a right angled triangle the largest angle is of 100°. **T /F**
   - (v) A perpendicular on a line always makes and angle of 90°. **T /F**

2. **Fill in the blanks to make the sentences true sentences.**
   - (i) In any right angled triangle, _______ is the longest side of the triangle.
   - (ii) In a right angled triangle, sum of the measures of the sides containing right angles is _______ than the measure of the hypotenuse.
   - (iii) In ΔABC, $m\angle A = 50^{\circ}$ and $m\angle B = 30^{\circ}$. Side _______ will be longer than its other sides.
   - (iv) Length of diagonal of any quadrilateral is ____________ than the sum of the measures of its two adjacent sides.

3. Tick (✓) the correct answer.

(i) Measure of one side of an equilateral triangle is 6 cm, then the length of its median is ______ 9cm.

a) less than b) greater than

c) equal to d) none of the above

(ii) Perimeter of a rectangle is 22cm, then the length of its diagonal is _____ 11cm.

a) equal to b) greater than

c) less than d) none of the above

Summary

- If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.
- If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
- From a point, outside a line, the perpendicular is the shortest distance from the point to the line.

Unit 13

• Weightage = 3%

PRACTICAL GEOMETRY – TRIANGLES

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Construct a triangle having given:
  - Two sides and the included angle,
  - One side and two of the angles,
  - Two of its sides and the angle opposite to one of them, (with all the three possibilities)
- Draw :
  - Angle bisectors,
  - Altitudes,
  - Perpendicular bisectors,
  - Medians of a given triangle and verify their given concurrency
- Construct a triangle equal in area to a given quadrilateral.
- Construct a rectangle equal in area to a given triangle.
- Construct a square equal in area to a given rectangle.
- Construct a triangle of equivalent area on a base of given length.

13.1 Construction of a Triangle

13.1.1 Construct a triangle having given:

- Two sides and included angles.
- One side and two angles.
- Two of its sides and the angle apposite to one of them. With all three possibilities.

When two sides and the included angle are given.

Example Construct a triangle ABC in which
$m\overline{AB} = 4.8\,\text{cm}$, $m\overline{AC} = 3.2\,\text{cm}$ and $m\angle B = 75°$

Construction:

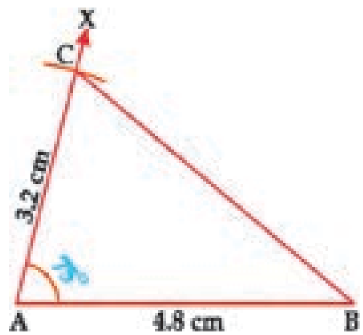
i) Draw the line segment $\overline{AB}$ of measure 4.8 cm.

ii) At point A, draw on angle XAB of measure 75°.

iii) Cut $\overline{AC}$ of measure 3.2 cm from $\overrightarrow{AX}$

iv) Draw $\overline{BC}$

Thus ΔABC is the required triangle.



When one side and two angles are given.

Example 01 Construct a triangle ΔABC in which
$m\overline{AB} = 5.5\,\text{cm}$, $m\angle A = 75°$ and $m\angle B = 40°$

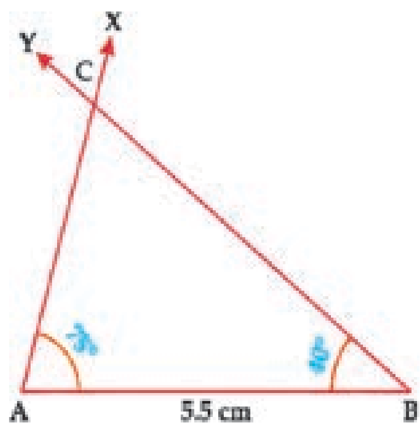
Construction:

i) Draw the line segment $\overline{AB}$ of measure 5.5 cm.

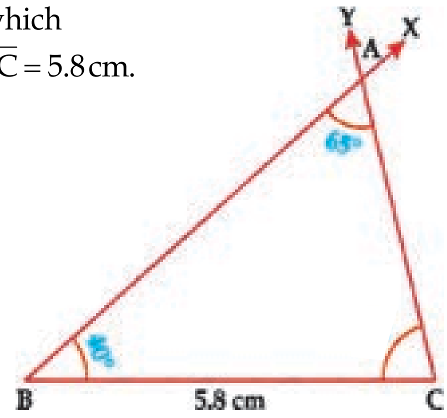
ii) At point A, draw on angle XAB of measure 75°.

iii) At point B draw an angle YBA of measure 40°, such that $\overrightarrow{BY}$ cuts $\overrightarrow{AX}$ at point C.

Thus ΔABC is the required triangle.



Example 02 Construct a triangle ΔXYZ in which
$m\angle A = 65^\circ$, $m\angle B = 40^\circ$ and $m\overline{BC} = 5.8\,\text{cm}$.



Construction:

We know that in ΔABC
$m\angle A + m\angle B + m\angle B = 180^\circ$
Here $m\angle A = 65^\circ$ and $m\angle B = 40^\circ$
So, $m\angle C = 180^\circ - (m\angle A + m\angle B)$
$= 180^\circ - (65^\circ + 40^\circ)$
$= 180^\circ - 105^\circ$
$= 75^\circ$

We now construct the triangle with
$m\overline{BC} = 5.8\,\text{cm}$, $m\angle B = 40^\circ$ and $m\angle C = 75^\circ$

i) Draw $\overline{BC}$ of measure 5.8 cm.
ii) Draw an angle $XBC = 40^\circ$, at point B.
iii) Draw $m\angle YCB = 75^\circ$ at point C.
iv) Rays $\overrightarrow{BX}$ and $\overrightarrow{CY}$ intersect each other at point A,
Thus ΔABC is the required triangle.

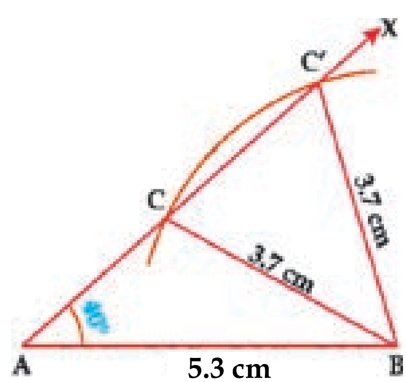
Two of its sides and the angle apposite to one of them. With all three possibilities.

Case I

Example 01 Construct a triangle ABC in which
$m\angle A = 40^\circ$, $m\overline{BC} = 3.7\,\text{cm}$ and $m\overline{AB} = 5.3\,\text{cm}$

Construction:

i) Draw $\overline{AB}$ of measure 5.3 cm.
ii) At point A, draw $\angle BAX$ of measure 40°.
iii) With center B and radius 3.7 cm, draw an arc which cuts $\overrightarrow{AX}$ at point C and C′.
iv) Draw $\overline{BC}$ and $\overline{BC'}$
ΔABC and $\Delta ABC'$ are the required triangle.



Case II

Example 02 Construct a triangle DEF when

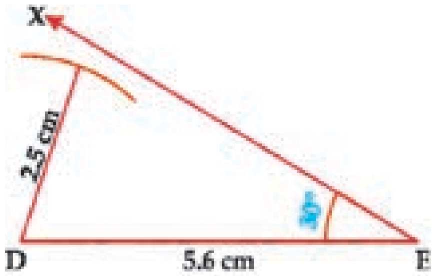
$m\overline{DE} = 5.6\,\text{cm},\ m\overline{DF} = 2.5\,\text{cm}$ and $m\angle E = 30^\circ$

Construction:

i) Draw a line segment $\overline{DE}$ of measure 5.6 cm.

ii) Draw and angle DEX of measure 30° at point E.

iii) With D as a center draw an arc of radius 2.5 cm, which does not cut $\overrightarrow{EX}$ at any point. In this case no triangle can be constructed satisfying the given data.



Case III

Example 03 Construct a triangle PQR when

$m\overline{PR} = m\overline{QR} = 4.7\,\text{cm}$ and $m\angle P = 55^\circ$

Construction:

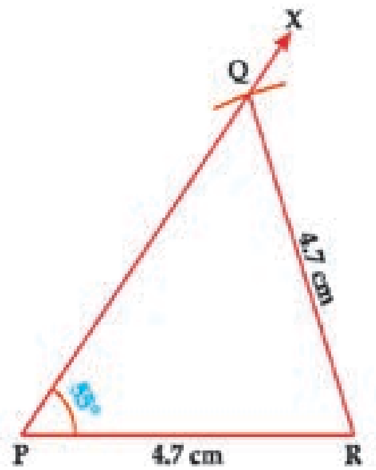
i) Draw a line segment $\overline{PR}$ of measure 4.7 cm.

ii) Draw an angle $\angle XPR$ of measure 55° at point P.

iii) With point R as a center draw an arc of radius 4.7 cm, which cuts $\overrightarrow{PX}$ at point Q.

iv) Join point Q and R.

ΔPQR is the required triangle.



Note: The above case I, case II and case III are called ambiguous cases.

Exercise 13.1

1. Construct ΔPQR such that, $m\overline{PQ} = m\overline{QR} = 4.6$ cm and $m\angle Q = 35^\circ$
2. Construct ΔABC such that, $m\overline{AB} = m\overline{AC} = 5.1$ cm and $m\angle A = 65^\circ$
3. Construct ΔLMN such that, $m\overline{LM} = 3.7$ cm, $m\overline{MN} = 2.5$ cm and $\angle M = 50^\circ$
4. Construct ΔABC such that, $m\overline{AB} = 3.5$ cm, $m\overline{BC} = 2.7$ cm and $\angle B = 110^\circ$
5. Construct ΔXYZ such that, $m\overline{XY} = 4.1$ cm, $m\overline{YZ} = 5$ cm and $\angle Z = 80^\circ$

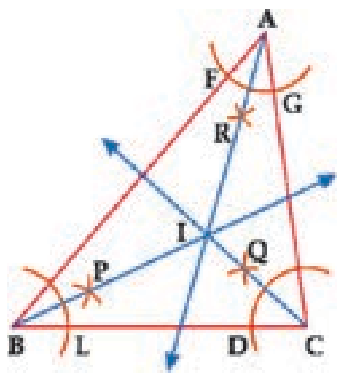
6. Construct the ΔDEF, ΔLMN and ΔABC in the following.
   (i) $m\overline{DE} = 5cm$, $m\angle D = 45°$ and , $m\angle E = 60°$
   (ii) $m\overline{LM} = 6cm$, $m\angle L = 75°$ and , $m\angle M = 45°$
   (iii) $m\overline{BC} = 5.8cm$, $m\angle A = 30°$ and , $m\angle B = 45°$
7. Construct a ΔABC, when lengths of two of its sides and measure of an angle opposite one of the side is given as under:
   (i) $m\overline{AC} = 4.5cm$, $m\overline{BC} = 4.1cm$ and $m\angle B = 75°$
   (ii) $m\overline{BC} = 5cm$, $m\overline{AB} = 5.5$ cm and $m\angle C = 70°$
   (iii) $mAB = 5cm$, $m\overline{BC} = 5.5cm$ and $m\angle A = 45°$

13.1.2 Draw

- **Angle bisectors**
- **Altitudes**
- **Perpendicular bisectors**
- **Medians**

(i) Draw the angle bisector of a given triangle

Example Draw bisectors of angle of ΔABC.



Given:

ABC is a triangle ∠A, ∠B and ∠C are its angles.

Required:

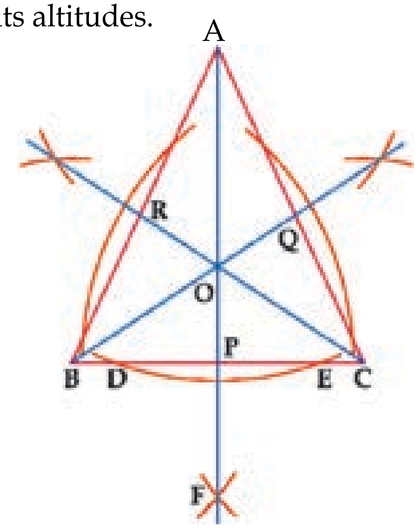
To draw bisectors of ∠A, ∠B and ∠C.

Construction:

i) Draw the triangle ABC.
ii) With point B as a center draw an arc of any radius, intersecting the sides $\overline{BC}$ and $\overline{BA}$ at points L and M.
iii) Take point L as a center and draw an arc of any radius.
iv) Now take point M as a center and with the same radius draw another arc, which cuts the pervious arc at point P.
v) Join point P to B and produce it.
   $\overrightarrow{BP}$ is the bisector of ∠B.
vi) Repeat steps (ii) to (v) to draw $\overrightarrow{CQ}$ and $\overrightarrow{AR}$ the bisectors of ∠C and ∠A respectively.
   Hence, $\overrightarrow{BP}$, $\overrightarrow{CQ}$, and $\overrightarrow{AR}$ are the required bisector of ΔABC.

(ii) Draw the altitudes of a given triangle

Example Take any triangle ABC and draw its altitudes.



Given:

A ΔABC

Required:

To draw altitudes of the ΔABC.

Construction:

i) Draw the triangle ABC.

ii) Take point A as center and draw an arc of suitable radius, which cuts $\overline{BC}$ at points D and E.

iii) From D as center, draw an arc of radius more than $\frac{1}{2}m\overline{DE}$.

iv) Again from point E draw another arc of same radius, cutting first arc at point F.

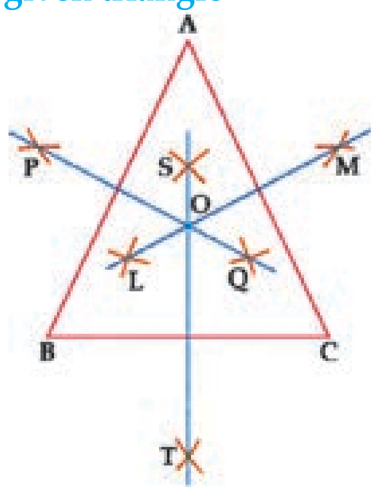
v) Join the points A and F. Such that $\overrightarrow{AF}$ intersects $\overline{BC}$ at point P. Then $\overline{AP}$ is the altitude of the ΔABC from the vertex A.

vi) Repeat the steps (ii) to (v) and draw $\overline{BQ}$ and $\overline{CR}$, the altitudes of ΔABC from the vertices B and C, respectively.

Hence $\overline{AP}$, $\overline{BQ}$ and $\overline{CR}$ are required altitudes of ΔABC

(iii) Draw the perpendicular bisector of a given triangle

Example Draw the perpendicular bisector of sides of a triangle ABC.



Given:

A triangle ABC.

Required:

To draw perpendicular bisectors of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$.

Construction:

i) Draw the triangle ABC.

ii) To draw perpendicular bisector of the side $\overline{AB}$, with B as a center and radius more than half of $\overline{AB}$, draw arcs on either sides of $\overline{AB}$.

iii) Now with A as a center and with the same radius, draw arcs on either sides of $\overline{AB}$, cutting previous arcs at P and Q.

iv) Join P and Q.

$\overline{PQ}$ is the perpendicular bisector of the $\overline{AB}$.

v) Repeat the steps (ii) to (iv) and draw $\overline{ST}$ and $\overline{LM}$, the perpendicular bisectors of $\overline{BC}$ and $\overline{AC}$, respectively.

Hence $\overline{PQ}$, $\overline{ST}$ and $\overline{LM}$ are the required perpendicular bisector of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$, respectively, of the ΔABC.

(iv) Draw the median of a given triangle

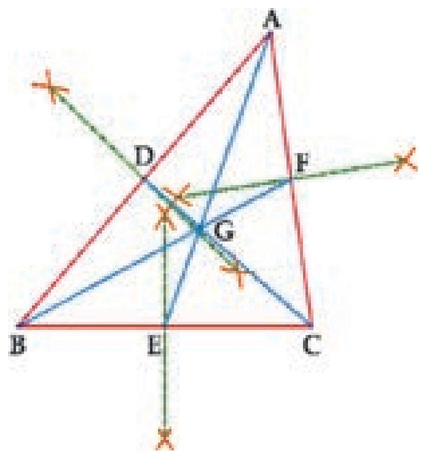
Example Take any triangle ABC and draw medians of this triangle.

Given:

A ΔABC

Required:

To draw medians of the ΔABC.



Construction:

i) Draw the triangle ABC.

ii) Bisect the sides $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ at points D, E and F, respectively.

iii) Join A to E; B to F and C to D.

Thus $\overline{AE}$, $\overline{BF}$ and $\overline{CD}$ are the required medians of the ΔABC, which meet in a point G.

It may be noted that medians of every triangle are concurrent (i.e., meet in one point) and their point of concurrency, called centroid, divides each of them in 2 : 1.

By actual measurement it can be proved that

$$\frac{m\overline{AG}}{m\overline{GE}} = \frac{m\overline{BG}}{m\overline{GF}} = \frac{m\overline{CG}}{m\overline{GD}} = \frac{2}{1}$$

Exercise 13.2

1. Take a Δ and draw the medians and prove that they are concurrent.
2. Take a Δ and draw the altitudes and prove that they are concurrent.
3. Take a Δ and draw the internal bisectors of angles and prove that they are concurrent.
4. Construct a triangle ABC in which $m\overline{BC}$ = 6 cm, $m\overline{CA}$ = 4 cm and $m\overline{AB}$ = 5 cm, draw the bisectors of angles A and B.
5. Construct a triangle PQR in which $m\overline{PQ}$ = 5.7cm, $m\overline{QR}$ = 6.4cm and $m\overline{PR}$ = 4.4 cm, draw the altitudes from vertex R and vertex Q.
6. Construct a triangle STU in which ∠T = 60°, ∠U = 30° and $m\overline{TU}$ = 7 cm. Find the perpendicular bisectors of the sides of triangle and prove that they are concurrent.
7. Construct a right triangle ABC in which ∠C = 90°, ∠B = 45° and $m\overline{CB}$ = 5 cm. Draw the medians of the triangle.
8. Construct the following ΔXYZ. Draw their three medians and show that they are concurrent.
   (i) $m\overline{YZ}$=4.4cm, m∠Y = 45° and m∠Z = 75°
   (ii) $m\overline{XY}$=4.6cm, m $m\overline{XZ}$ = 4.6cm and m∠Y = 60°
9. Construct the ΔKLM, in which $m\ \overline{KL}$ =4.8cm, $m\ \overline{LM}$ = 5.2cm and $m\overline{MK}$=4.5cm, draw their altitudes and verify their concurrency.
10. Construct the ΔPQR, in which $m\ \overline{PQ}$ =7cm, $m\ \overline{QR}$ = 6.5cm and $m\ \overline{PR}$ =5.8cm, find their perpendicular bisectors and verify their concurrency.

13.2 Figures with equal Areas

13.2.1 Construct a triangle equal in area to a given quadrilateral.

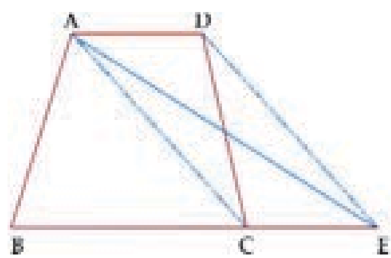
E.g. draw a triangle equal in area to given quadrilateral ABCD. We know that, Area of all triangles with same base equal of vertices are on the line perpendicular to base.

1. ABCD is a given quadrilateral.

2. Join A to C.

4. Through D, draw $\overline{DE}$ parallel to $\overline{AC}$ meeting $\overline{BC}$ produced at point E.

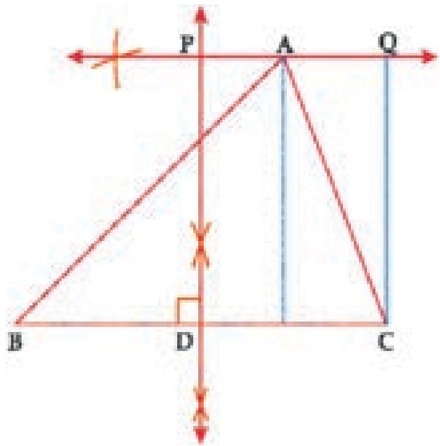
5. Join A to E, then ABE is the required triangle.



13.2.2 Construct a rectangle equal in area to a given triangle.

E.g. Construct a rectangle equal in area to given ΔABC

1. Draw a triangle ABC.
2. Draw a perpendicular bisector $\overleftrightarrow{PD}$ of $\overline{BC}$.
3. Through A, draw a line PQ parallel to $\overline{BC}$.
4. Take $m\overline{PQ} = m\overline{DC}$.
5. Then CDPQ is the required rectangle.

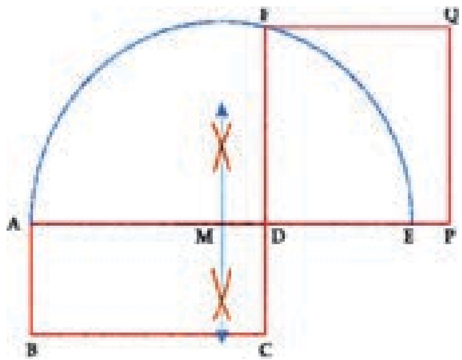


13.2.3 Construct a Square equal in area to a given rectangle.

E.g. Construct a square equal in Area to given rectangle ABCD.

Following are the steps of construction

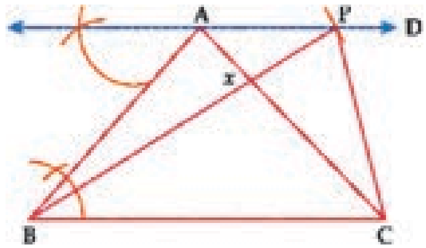
1. ABCD is a given rectangle.
2. Produce side $\overline{AD}$ to E making $m\overline{DE} = m\overline{CD}$.
3. Bisect $\overline{AE}$ at M.
4. With centre M and radius $m\overline{AM}$ construct a semi circle.
5. Produce $\overline{CD}$ to meet the semi circle at F.
6. On $\overline{DF}$ as a side construct a square DFQP. This shall be required square.



13.2.4 Construct a Triangle of equivalent area on a base of given length:

Following are the steps of construction

1. ABC is given triangle.
2. Draw $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.
3. With B as centre, and radius = x, such that $m\overline{BC} = x$ draw an arc cutting $\overleftrightarrow{AD}$ at P.
4. Join $\overline{BP}$ and $\overline{CP}$.
5. Then ΔBCP is the required triangle with equal base $\overline{BP} = x$ and area equivalent to ΔABC.



Exercise 13.3

1. Construct a rectangle whose adjacent sides are 2.5 cm and 5 cm respectively. Construct a square having area equal to the given rectangle.
2. Construct a square equal in area to a rectangle whose adjacent sides are 4.5 cm and 2.2 cm respectively. Measure the sides of the square and find its area and compare with the area of the rectangle. Also verify by measurement that the perimeter of the square is less than that of the rectangle.
3. Construct a triangle having base 4 cm and other two sides equal to 3.6cm and 3.8 cm each. Transform it into a rectangle with equal Area.
4. Construct a triangle having base 6cm and other sides equal to 5cm and 6cm each. Construct a rectangle equal in area to given Δ.

Review Exercise 13

1. **Fill in the blanks.**
   i) The side of a right triangle opposite to the right angle is __________
   ii) The line segment joining a vertex of a triangle and perpendicular to its opposite side is called an__________
   iii) A line segment drawn from a vertex of a triangle and meeting the mid-point of its opposite side is called a __________

iv) The perpendicular bisectors of the sides of a triangle are _________

v) Two or more than two triangles are said to be congruent if they are equiangular and measures of their corresponding sides are ______

2. Tick (✓) the correct answer.

i) A triangle having all the three sides congruent is called __________ triangle.

(a) scalene (b) right angled
(c) equilateral (d) isosceles

ii) A quadrilateral having each angle equal to 90° and all the sides are congruent is called __________

(a) parallelogram (b) rectangle
(c) trapezium (d) square

iii) The medians of a triangle are __________

(a) collinear (b) congruent
(c) concurrent (d) parallel

iv) The __________ altitudes of an equilateral triangle are congruent.

(a) two (b) three
(c) four (d) none

v) The diagonals of a rectangle __________ each other.

(a) bisect (b) trisect
(c) bisect at right angle (d) none of these

vi) The __________ of a triangle cut each other in the ratio 2:1.

(a) Altitudes (b) Angle bisectors
(c) Right bisectors (d) Medians

vii) If each angle on the base of an isosceles triangle is 45°, then the measure of the third angle is __________

(a) 30° (b) 60°
(c) 90° (d) 120°

viii) If the three medians of a triangle are congruent then the triangle is ____________

(a) Right angled (b) equilateral
(c) Isosceles (d) acute angled

ix) If two __________of a triangle are congruent, then the triangle will be isosceles.

(a) Altitudes (b) Medians
(c) Right bisectors (d) sides

Summary

- In this unit we have learnt the construction of the following figures and relevant concepts.
- To construct a triangle, having given two sides and the included angle.
- To construct a triangle, having given one side and two of the angles.
- To construct a triangle, having given two of its sides and the angle opposite to one of them.
- To draw angle bisectors of a given triangle and to verify their concurrency.
- To draw altitudes of a given triangle and verify their concurrency.
- To draw perpendicular bisectors of the sides of a given triangle and to verify their concurrency.
- To draw medians of a given triangle and verify their concurrency.
- To construct a triangle equal in area to a given quadrilateral.
- To construct a rectangle equal in area to given triangle.
- To construct a square equal in area to given rectangle.
- To construct a triangle of equivalent area on the base of given length.
- Two or more than three lines are said to be concurrent if these passes through a common point and that point is called the point of concurrency.
- The point where the internal bisectors of the angles of a triangle intersect is called the in-centre of a triangle.
- The point of concurrency of the perpendicular bisectors of the sides of a triangle is called its circum-centre.
- Median of a triangle means a line segment joining a vertex of a triangle to the mid-point of the opposite side.
- Ortho-centre of a triangle means the point of concurrency of three altitudes of a triangle.

Unit 14

• Weightage = 5%

THEOREMS RELATED WITH AREA

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Understand the following theorems along with their corollaries and apply them to solve allied problems.
- Parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.
- Parallelograms on equal bases and having the same altitude are equal in area.
- The triangles on the same base and of the same altitude are equal in area.
- Triangles on equal bases and of the same altitude are equal in area.

Introduction

We will study the theorems related with Area.

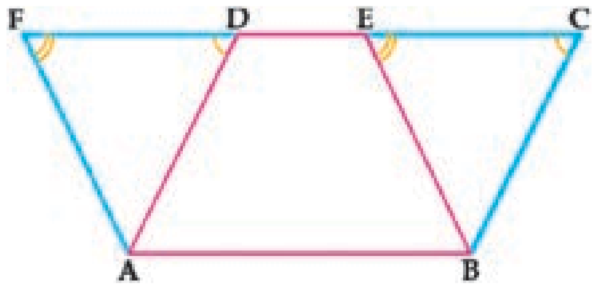
14.1 Theorems Related with Area

Theorem 14.1.1

Parallelograms on the same base and lying between the same parallel lines (or of the same altitude) are equal in area.

Given:

Two parallelograms ABCD and ABEF with the same base $\overline{AB}$ and between the same parallels segments $\overline{AB}$ and $\overline{DE}$.



To prove:

Parallelograms ABCD and ABEF are equal in areas,
i.e. ▰ABCD = ▰ABEF.

Proof:

Statements	Reasons
In ΔBCE ↔ ΔADF	
(i) $m\overline{BC} = m\overline{AD}$ ... (i)	(i) Opposite sides of $\|^{m}$ ABCD are equal.
(ii) $m\angle BCE = m\angle ADF$ ... (ii)	(ii) Corresponding angles of $\|^{m}$ ABCD.
(iii) $\angle E \cong \angle F$ ... (iii)	(iii) Corresponding angles of $\|^{m}$ ABEF.
∴ ΔBCE ≅ ΔADF	S.A.A ≅ S.A.A
∴ ▲BCE ≅ ▲ADF	Congruent figures are equal in area.
▰ABED + ▲BCE = ▰ABED + ▲ADF	Adding same area on both sides
Thus, ▰ABCD = ▰ABEF.	▰ABCD = ▰ABED + ▲BCE
	▰ABEF = ▰ABED + ▲ADF

Q.E.D

Corollary

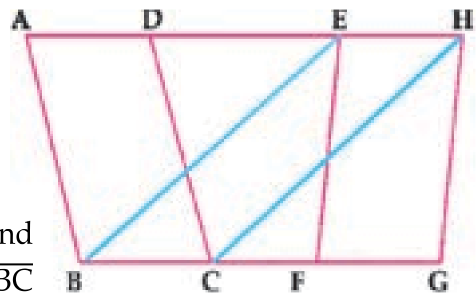
(i) The area of parallelogram is equal to that of a rectangle on the same base and having the same altitude.

Theorem 14.1.2

Parallelograms on equal bases and having the same altitude are equal in area.

Given:

Parallelograms ABCD and EFGH are on the equal bases $\overline{BC}$ and $\overline{FG}$, having equal altitudes.



To Prove: ▰ABCD = ▰EFGH.

Construction:

Place the parallelograms ABCD and EFGH so that their equal bases $\overline{BC}$ and $\overline{FG}$ are on the same straight line. Join B to E and C to H.

Proof:

Statements	Reasons
$\|^{m}$ ABCD and $\|^{m}$ EFGH are between the same parallel segments $\overline{AH}$ and $\overline{BG}$. Hence, A, D, E and H are points lying on a straight line parallel to $\overline{BC}$.	Their altitudes are equal (given)
$m\overline{BC} = m\overline{FG}$	Given
$m\overline{BC} = m\overline{EH}$	EFGH is a parallelogram and $m\overline{BC} = m\overline{FG}$
$m\overline{BC} = m\overline{EH}$ also these are parallel	Segment of parallel lines are also parallel segments.
Hence, EBCH is a parallelogram	A quadrilateral with two parallel opposite sides is a parallelogram
Now ▰ABCD = ▰EBCH … (i)	Theorem 14.1.1
But ▰EBCH = ▰EFGH … (ii)	Theorem 14.1.1
Thus, ▰ABCD = ▰EFGH	From (i) and (ii)

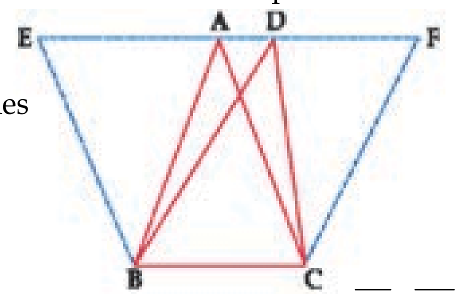
Q.E.D

Theorem 14.1.3

Triangles on the same base and of the same altitude are equal in area.

Given:

ΔABC and ΔDBC are on the same base $\overline{BC}$ and between the same parallel lines $\overline{BC}$ and $\overline{AD}$.



To prove:

▲ABC = ▲DBC

Construction:

Draw $\overline{BE} \parallel \overline{CA}$, meeting at $\overline{AD}$ produced, at E and also draw $\overline{CF}$ $\overline{BD}$ meeting at $\overline{AD}$ produced at F.

Proof :

Statements	Reasons
BCAE is a parallelogram.	By construction
▲ABC = $\frac{1}{2}$ (▰BCAE) … (i)	Diagonal $\overline{AD}$ divides parallelogram BCAE into two Δs of equal areas.
Similarly BCFD is a parallelogram	By construction
▲DBC = $\frac{1}{2}$ (▰BCFD) … (ii)	Diagonal $\overline{CD}$ divides parallelogram BCFD into two triangles of equal areas.
▰BCAE = ▰BCFD … (iii)	Theorem 14.1.1
▲ ABC = ▲DBC	From (i),(ii) and (iii)

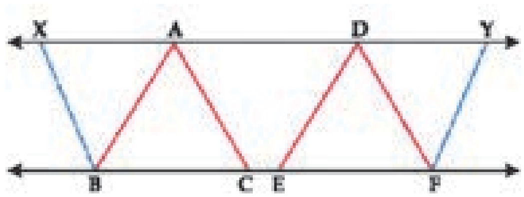
Q.E.D

Theorem 14.1.4

Triangles on equal bases and of equal altitudes are equal in area.

Given:

ΔABC and ΔDEF are on equal bases $\overline{BC}$ and $\overline{EF}$ respectively and having equal altitudes.



To prove:

▲ABC = ▲DEF

Construction:

Draw $\overleftrightarrow{AD}$, $\overleftrightarrow{BF}$ containing points B, C, E, F.

Place the ΔABC and ΔDEF so that their equal bases $\overline{BC}$ and $\overline{EF}$ are on the straight line. Draw $\overline{BX} \parallel \overline{CA}$ and $\overline{FY} \parallel \overline{ED}$. Such that point X and Y lie on $\overleftrightarrow{AD}$.

Proof :

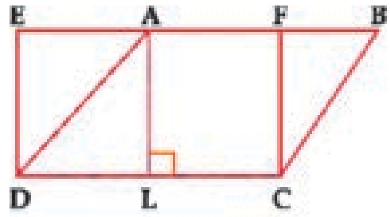
Statements	Reasons
ΔABC and ΔDEF are between the same parallel lines.	Altitudes are equal (given)
$\overleftrightarrow{BF} \parallel \overleftrightarrow{XY}$	construction
∴ ▰BCAX = ▰EFYD … (i)	Theorem 14.1.2
But, ▲ABC $= \frac{1}{2}$ (▰BCAX) … (ii)	Diagonal of a parallelogram divides $\|^{m}$ into two equal triangles
and ▲DEF $= \frac{1}{2}$ (▰EFYD) … (iii)	By same reason
∴ ▲ABC = ▲DEF.	From eqs.(i), (ii) and (iii)

Q.E.D

Corollary: Triangles having a common vertex and equal bases in the same straight line are equal in area.

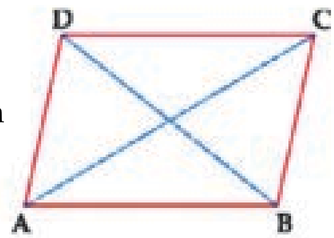
Exercise 14.1

1. In the given figure, ABCD is a parallelogram and EFCD is a rectangle, also $\overline{AL} \perp \overline{DC}$. Prove that

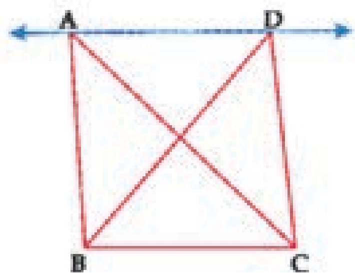


   (i) ▰ABCD = ▰EFCD

   (ii) ▰ABCD = $m\overline{DC} \times m\overline{AL}$.



2. In the given figure, if the diagonals of a quadrilateral separate it into four triangles of equal area, show that it is a parallelogram.



3. In the given figure $\overline{BC} \parallel \overline{AD}$. ABC is a right-angled at vertex B with $m\overline{BC} = 7$ cm and $m\overline{AC} = 11$ cm, also ΔABC and ΔBCD are on the same base $\overline{BC}$. Find the area of ΔBCD.

4. Show that a median of a triangle divides it into two triangles of equal area.

5. Show that the line segment joining the mid-points of the opposite sides of a rectangle, divides it into two equal rectangles.

6. If two parallelograms of equal areas have the same or equal bases, their altitudes are equal.

7. Show that an angle bisector of an equilateral triangle divides it into two triangles of equal areas.

8. Prove that a rhombus is divided by its diagonals into four triangles of equal areas.

Review Exercise 14

1. **Mark 'T' for True and 'F' for False in front of each given below:**
   - (i) Area of a closed figure means region enclosed by bounding lines of the figure. **T/F**
   - (ii) A diagonal of rectangle divides it into two congruent triangles. **T/F**
   - (iii) Congruent figures have different areas. **T/F**
   - (iv) The area of parallelogram is equal to the product of base and height. **T/F**
   - (v) Median of a triangle means perpendicular from a vertex to the opposite side (base). **T/F**
   - (vi) Perpendicular distance between two parallel lines can sometimes be different. **T/F**
   - (vii) An altitude drawn from a vertex always bisects the opposite side. **T/F**
   - (viii) Two triangles are equal in areas, if they have the same base and equal altitude. **T/F**

2. **Tick (✓) the correct answer.**
   - (i) If perpendicular distance between two lines is the same. The lines are ______________
     - (a) Perpendicular to each other (b) Parallel to each other
     - (c) Intersecting to each other (d) None of these.
   - (ii) If two triangles have equal areas then they will _____________be congruent as well.
     - (a) Not necessarily (b) Necessarily
     - (c) Definitely (d) None of these.
   - (iii) Perpendicular from a vertex of a triangle to its opposite side is called ______________.
     - (a) Median (b) Perpendicular bisector
     - (c) Altitude (d) Angle bisector

(iv) Parallelograms having same base and same altitude are _____.
(a) Congruent (b) Equal in areas
(c) Similar (d) All of these.

(v) Two parallelograms have equal bases. They will be having the same area, if _____________.
(a) Their altitudes are equal
(b) Their altitude is the same
(c) They lies between the same parallel lines
(d) All of these.

(vi) ΔABC and ΔDEF have equal bases and equal altitudes, then triangles are_____________.
(a) Equal in area (b) Congruent
(c) Similar (d) None of these.

Summary

- In this unit we have mentioned some necessary preliminaries, stated and proved the following theorems along with corollaries, if any.
- Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in areas.
- Parallelograms on the equal bases and having the same (or equal) altitude are equal in areas.
- Triangles on the same base and of the same (i.e. equal) altitudes are equal in areas.
- Triangles on equal bases and of equal altitudes are equal in areas.
- Area of a figure means region enclosed by the boundary lines of a closed figure.
- A triangular region means the union of triangle and its interior.
- By area of triangle means the area of its triangular region.
- Altitude or height of a triangle means perpendicular distance to base from its opposite vertex.

Unit 15

• Weightage = 7%

PROJECTION OF A SIDE OF A TRIANGLE

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- In an obtuse–angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half of the third side together with twice the square on the median which bisects the third side, (Apollonius' theorem).

15.1 Projection of a side of a Triangle

Understand the following theorems along with their corollaries and their applications to solve the allied problems.

Theorem 15.1.1

In an obtuse–angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.

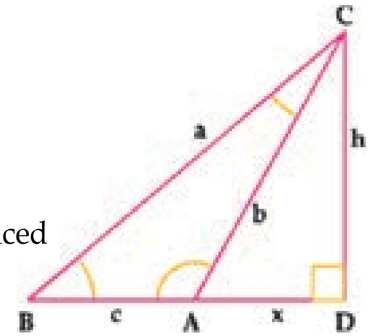
Given:

ΔABC with an obtuse angle at vertex A.

To prove:

i.e. $a^2 = b^2 + c^2 + 2cx$

Construction:



Draw perpendicular $\overline{CD}$ on $\overline{BA}$ produced meeting at point D, so that $\overline{AD}$ is the projection of $\overline{AC}$ on $\overline{BA}$ produced.

Taking, $m\overline{BC} = a$, $m\overline{CA} = b$, and $m\overline{AB} = c$ also $m\overline{AD} = x$ and $m\overline{CD} = h$.

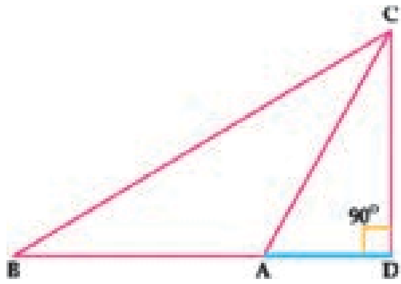
Proof:

Statements	Reasons
In right angled ΔCDA	
$m\angle CDA = 90^0$	Construction
so, $(m\overline{AC})^2 = (m\overline{AD})^2 + (m\overline{DC})^2$	By Pythagoras theorem.
i.e $b^2 = x^2 + h^2$ ... (i) ,	By supposition
In right angled ΔCDB	
$m\angle CDA = 90^0$	Construction
so, $(m\overline{BC})^2 = (m\overline{BD})^2 + (m\overline{CD})^2$	By Pythagoras theorem.
i.e. $a^2 = (c+x)^2 + h^2$	$m\overline{BD} = m\overline{BA} + m\overline{AD}$
or $a^2 = c^2 + 2cx + x^2 + h^2$ ... (ii)	
$a^2 = c^2 + 2cx + b^2$	$b^2 = x^2 + h^2$
Thus $(m\overline{BC})^2 = (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD}) + (m\overline{AC})^2$	

Q.E.D

Example: In a ΔABC with obtuse angle is at vertex A, if $\overline{CD}$ is an altitude on $\overline{BA}$ produced, meeting at point D, and $m\overline{AC}=m\overline{AB}$

Then, $\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)\left(m\overline{BD}\right)$.



Given:

In a ΔABC, $m\angle A$ is an obtuse, $m\overline{AC} = m\overline{AB}$ and $\overline{CD}$ being altitude on $\overline{BA}$ produced, meeting at point D.

To prove:

$$\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)\left(m\overline{BD}\right).$$

Proof:

Statements	Reasons
In a ΔABC	
$\left(m\overline{BC}\right)^2 = \left(m\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right) + \left(m\overline{AC}\right)^2$	Theorem 15.1.1
$\left(m\overline{BC}\right)^2 = \left(m\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right) + \left(m\overline{AB}\right)^2$	Given that $m\overline{AC} = m\overline{AB}$
$\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)^2 + 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$	Taking $2m\overline{AB}$ as common
$\left(m\overline{BC}\right)^2 = 2m\overline{AB}\left(m\overline{AB} + m\overline{AD}\right)$	$m\overline{BD} = m\overline{AD} + m\overline{AB}$
$\left(m\overline{BC}\right)^2 = 2\left(m\overline{AB}\right)\left(m\overline{BD}\right)$	

Q.E.D

Exercise 15.1

1. Find the length of $\overline{AB}$ and area of the triangle ABC, when

(i) $m\overline{AC} = 3cm,\ m\overline{BC} = 6\ cm$ and $m\angle C = 120^\circ$, where $m\overline{CD} = m\overline{BC}\ \cos(180^\circ - m\angle C)$

(ii) $m\overline{AC} = 40\,mm,\ m\overline{BC} = 80\ mm$ and $m\angle C = 120^\circ$, where $m\overline{CD} = m\overline{BC}\ \cos(180^\circ - m\angle C)$

Hint: $(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 + 2(m\overline{AB})(m\overline{AD})$

2. Find the length of $m\overline{AC}$ in the ΔABC when $m\overline{BC} = 6cm$, $m\overline{AB} = 4\sqrt{2}\ cm$ and $m\angle ABC = 135^\circ$,. If possible, find the area of the ΔABC.

3. Find the length of $m\overline{AC}$ in the ΔABC when $m\overline{BC} = 6\sqrt{2}\,cm$, $m\overline{AB} = 8\ cm$ and $m\angle ABC = 135^\circ$,. If possible, find the area of the ΔABC.

Theorem 15.1.2

In any triangle, the square on the side opposite to an acute angle is equal to sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.

Given:

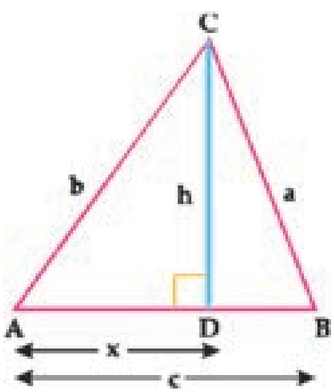
ΔABC with an acute $\angle CAB$ at vertex A.

Taking, $m\overline{BC} = a,\ m\overline{AC} = b$ and $m\overline{AB} = c$

Construction:

Draw $\overline{CD} \perp \overline{AB}$ so that $\overline{AD}$ is projection of $\overline{AC}$ on $\overline{AB}$

Also $m\overline{AD} = x$ and $m\overline{CD} = h$



To prove:

$$(m\overline{BC})^2 = (m\overline{AC})^2 + (m\overline{AB})^2 - 2(m\overline{AB})(m\overline{AD})$$

i.e. $a^2 = b^2 + c^2 - 2cx$

Proof:

Statements	Reasons
In right angled ΔCDA,	
$m\angle CDA = 90^0$,	Construction
so, $\left(m\overline{AC}\right)^2 = \left(m\overline{AD}\right)^2 + \left(m\overline{CD}\right)^2$	Using Pythagoras Theorem.
i.e. $b^2 = x^2 + h^2$ … (i)	By supposition
In right angled ΔCDA	
$m\angle CDB = 90^0$	Construction
$\left(m\overline{BC}\right)^2 = \left(m\overline{BD}\right)^2 + \left(m\overline{CD}\right)^2$	By Pythagoras Theorem.
so, i.e.	From the figure.
$a^2 = (c-x)^2 + h^2$	$\therefore\ m\overline{BD} = m\overline{AB} - m\overline{AD}$
$a^2 = c^2 - 2cx + x^2 + h^2$	
or $a^2 = c^2 - 2cx + b^2$ … (ii)	Using equation... (i)
$a^2 = c^2 + b^2 - 2cx$	
Hence	
$\left(m\overline{BC}\right)^2 = \left(m\overline{AC}\right)^2 + \left(m\overline{AB}\right)^2 - 2\left(m\overline{AB}\right)\left(m\overline{AD}\right)$	

Q.E.D

Apollonius and the theorem of Apollonius:

Apollonius was a great geometer and astronomer.

Now we state and prove one of his well-known theorem "the Apollonius theorem".

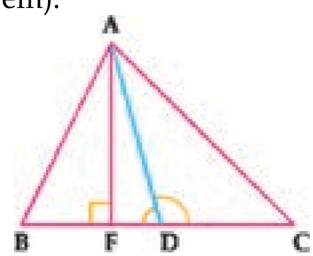
Theorem 15.1.3 (Apollonius theorem)

In any triangle, the sum of the squares on any two sides is equal to twice the square on half of the third side together with twice the square on the median which bisects the third side, (Apollonius' theorem).

Given:

In ΔABC, the median $\overline{AD}$ bisects $\overline{BC}$ at point D.

such that $m\overline{BD} = m\overline{CD}$.



To prove:

$$(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{BD})^2 + 2(m\overline{AD})^2$$

Construction:

Draw $\overline{AF} \perp \overline{BC}$.

Proof:

Statements	Reasons
In ΔADB	
Since, $\angle ADB$ is acute	ΔADF is right angled triangle with right angled at F (construction)
So, $(m\overline{AB})^2 = (m\overline{BD})^2 + (m\overline{AD})^2 - 2(m\overline{BD})(m\overline{FD})$...(i)	Using theorem 15.1.2
Now, In ΔADC	
We $\angle ADC$ is an obtuse angle at point D.	Supplement of in $\angle ADB$
So, $(m\overline{AC})^2 = (m\overline{CD})^2 + (m\overline{AD})^2 + 2(m\overline{CD})(m\overline{FD})$	Theorem 15.1.1
$(m\overline{AC})^2 = (m\overline{BD})^2 + (m\overline{AD})^2 + 2(m\overline{BD})(m\overline{FD})$...(ii)	$m\overline{CD} = m\overline{BD}$
Thus, $(m\overline{AB})^2 + (m\overline{AC})^2 = 2(m\overline{BD})^2 + 2(m\overline{AD})^2$	Adding eqs. (i) and (ii)

Q.E.D

Exercise 15.2

1. In $\Delta ABC, m\angle A = 30^0$ m, prove that
$\left(m\overline{BC}\right)^2 = \left(m\overline{AC}\right)^2 + \left(m\overline{AB}\right)^2 - \sqrt{3}\left(m\overline{AB}\right)\left(m\overline{AC}\right)$.

2. In a ΔABC, calculate $m\overline{BC}$ when $m\overline{AB} = 6cm, m\overline{AC} = 5cm$, and $m\angle A = 60^0$

3. Whether the triangle with sides 3cm, 4cm and 5cm is acute, obtuse or right angled.

4. Find the length of the median of side $\overline{BC}$ of a ΔABC where $m\overline{AB}$ = 4cm, $m\overline{AC}$ = 3cm and $m\overline{BC}$ = 6cm.

Review Exercise 15

1. **Fill in the blanks.**

(i) In rt. $\Delta ABC, \left(m\overline{AB}\right)^2 +$ __________ $= \left(m\overline{AC}\right)^2$,

(ii) In ΔABC, two sides are equal to 4cm it is called__________ triangle.

(iii) In ΔABC, with $m\angle B = 90^0$ then
$\left(m\overline{AB}\right)^2 + \left(m\overline{BC}\right)^2 =$ _________

(iv) 8cm, 15cm and 17 cm are the sides of ____________

2. In ΔABC, $m\overline{AC} = 3cm, m\overline{BC} = 6cm$ and $m\angle C = 120^0$. Compute $m\overline{AB}$

Summary

- In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.
- In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangle contained by one of those sides and the projection on it of the other.
- In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side, (Apollonius theorem).

Unit 16

• Weightage = 5%

Introduction To Coordinate Geometry / Analytical Geometry

Student Learning Outcomes (SLOs)

After completing this unit, students will be able to:

- Explain and define coordinate geometry.
- Derive distance formula to calculate distance between two points given in Cartesian plane.
- Use distance formula to find distance between two given points.
- Define collinear points. Distinguish between collinear and non- collinear points.
- Use distance formula to show that three (or more) given points are collinear.
- Use distance formula to show that the given three non-collinear points form:
  - i. An equilateral triangle,
  - ii. An isosceles triangle,
  - iii. A right angled triangle,
  - iv. A scalene triangle.
- Use distance formula to show that four given non- collinear points form:
  - A square,
  - A rectangle,
  - A parallelogram.
- Recognize the formula to find the midpoint of the line joining two given points.
- Apply distance and midpoint formulas to solve/verify different standard results related to geometry.

Introduction

The Cartesian coordinate was invented in the 17th century by Rene Descartes (Latinized name as Cartesius) revolutionized mathematics by providing the first systematic link between Euclidean geometry and algebra. Using the Cartesian co-ordinate system, geometric shapes (such as lines and curves) can be described by equations.

16.1 Distance Formula

16.1.1 Explain and define Coordinate Geometry

Co-ordinate geometry is one of the most important and exciting branch of mathematics. In particular it is central to the mathematics students meet at school. It provides a connection between algebra and geometry through graphs of lines and curves.

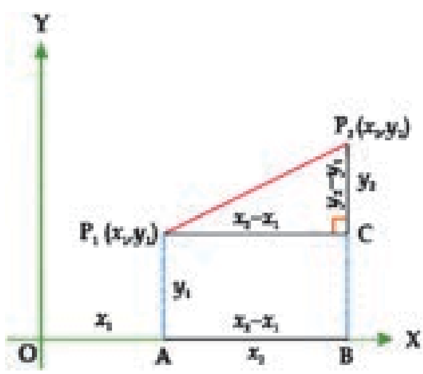
The algebraic study of geometry with the help of coordinate system is called co-ordinate geometry/analytical geometry.
This enables geometrical problems to be solved algebraically and provides geometric insights into algebra. It is a part of geometry in which ordered pairs of numbers are used to describe the position of a point on a plane. Here, the concept of coordinate geometry (also known as Cartesian geometry) and its formulas and their derivations will be explained.

16.1.2 Derive distance formula to calculate the distance between two given points in the Cartesian plane

Statement:

The distance between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is denoted as $|\overline{P_1P_2}|$ and is defined as:

$$|\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Derivation of the Distance Formula

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the any two points in plane.

From P_1 and P_2 draw perpendiculars $\overline{P_1A}$ and $\overline{P_2A}$ on x-axis, Also draw a $\overline{P_1C}$ parallel to x-axes.

$\left|\overline{P_1C}\right| = \left|\overline{AB}\right| = \left|\overline{OB}\right| - \left|\overline{OA}\right| = |x_2 - x_1|$

and $\left|\overline{P_2C}\right| = \left|\overline{P_2B}\right| - \left|\overline{BC}\right| = |y_2 - y_1|$

Consider right angled ΔP_1CP_2 and Appling Pythagoras theorem, we have,

$$\therefore \quad \left|\overline{P_1P_2}\right|^2 = \left|\overline{P_1C}\right|^2 + \left|\overline{P_2C}\right|^2$$

$$\Rightarrow \quad \left|\overline{P_1P_2}\right|^2 = (x_2 - x_1)^2 + (y_2 - y_2)^2$$

$$\Rightarrow \quad \left|\overline{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2}$$

Note:

The distance d from origin to the point $P(x, y)$ is:

$$d = \sqrt{x^2 + y^2}$$

16.1.3 Use Distance Formula to find the distance between two given points.

The following examples will help to understand the use of distance formula.

Example 01 Find the distance between the point $P(2,3)$ and $Q(-4,5)$

Solution: By using distance formula $\left|\overline{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Here

$$(x_1, y_1) = (2,3), (x_2, y_2) = (-4,5)$$

$$\therefore \quad \left|\overline{PQ}\right| = \sqrt{(-4-2)^2 + (5-3)^2}$$

$$\Rightarrow \quad \left|\overline{PQ}\right| = \sqrt{(-6)^2 + (2)^2} = \sqrt{36+4}$$

$$\Rightarrow \quad \left|\overline{PQ}\right| = \sqrt{40} = 2\sqrt{10}$$

Example 02 Circle with radius 5 unit is drawn with centre $C(3,2)$ and $L(6,6), M(0,-1)$ and $N(-2,-3)$ points are given. Find which of the point is not on the circle. (give reason).

Solution:

$C(3,2)$ is the centre of a circle with radius 5 units.

and *L* (6, 6), *M* (0, -1) *N* (-2, -3) are three given points.

We know that,

We have to, find the length (distance) from *C to L, M* and *N* respectively using distance formula, then,

$$\therefore \quad |\overline{CL}| = \sqrt{(6-3)^2+(6-2)^2} = \sqrt{9+16} = \sqrt{25} = 5 \text{ units,}$$

$$|\overline{CM}| = \sqrt{(0-3)^2+(-1-2)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2} \text{ units,}$$

$$|\overline{CN}| = \sqrt{(-2-3)^2+(-3-2)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2} \text{ units,}$$

$|\overline{CL}| = 5$ units i.e, radius of the circle

$|\overline{CM}| = 3\sqrt{2}$ units $<$ 5 units

$|\overline{CN}| = 5\sqrt{2}$ units $>$ 5 units

So, and $N(-2,-3)$ are not on the circle.

Exercise 16.1

1. Using distance formula find the distance between the following pairs of points.

(i) $(-4,5)$ and $(6,6)$ (ii) $(2,2)$ and $(2,3)$

(iii) $(0,1)$ and $(2,3)$ (iv) $(0,1)$ and $(2,3)$

2. $A(a,0)$ and $B(0,b)$ be the points on the axes, find the distance between A and B, when

(i) $a=-3,\ b=-4$ (ii) $a=-9,\ b=6$

(iii) $a=3,\ b=4$ (iv) $a=\sqrt{2},\ b=-2\sqrt{2}$

3. Find the perimeter of the triangle formed by the points. $A(0,0), B(4,0)$ and $C(2,2\sqrt{3})$.

16.2 Collinear Points

16.2.1 Define collinear points. Distinguish between collinear and non- collinear points.

Collinear Points

Definition:

Three or more points are said to be collinear if they lie on the same line.

In the following figure

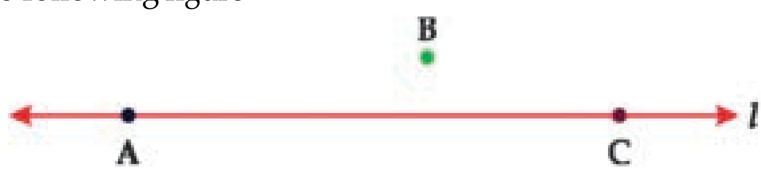
Points A, B and C are collinear points i.e. $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$

Non-Collinear Points

Definition:

Three or more points are said to be non-collinear points, if they do not lie on same line.

In the following figure



Points A, B and C are non-collinear points.

Notes:

1. Three non-collinear points form a triangle and four non-collinear points form a quadrilateral.

2. If points P, Q and R are collinear, then either

(i) $|\overline{PR}| = |\overline{PQ}| + |\overline{QR}|$

or (ii) $|\overline{PQ}| = |\overline{PR}| + |\overline{RQ}|$

or (iii) $|\overline{QR}| = |\overline{QP}| + |\overline{PR}|$ holds good,

otherwise the points are non-collinear.

16.2.2 Use Distance Formula to show that three (or more) given points are collinear.

Example 01 Show that the points $A(3,-2), B(1,4)$ and $C(-3,16)$ are collinear points.

Solution:

$A(3,-2), B(1,4)$ and $C(-3,16)$ are three given points.

Now we find the distances $|\overline{AB}|, |\overline{BC}|$ and $|\overline{AC}|$ by using distance formula.

$$\therefore \quad |\overline{AB}| = \sqrt{(1-3)^2+(4+2)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10} \text{ units,}$$

$$|\overline{BC}| = \sqrt{(-3-1)^2+(16-4)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10} \text{ units,}$$

$$|\overline{AC}| = \sqrt{(-3-3)^2+(16+2)^2} = \sqrt{36+324} = \sqrt{360} = 6\sqrt{10} \text{ units,}$$

Here, $|\overline{AB}|+|\overline{BC}| = 2\sqrt{10}+4\sqrt{10} = 6\sqrt{10} = |\overline{AC}| = d_3$,

Therefore the points A, B and C are collinear points. Showed.

Example 02 Using distance formula show that $A(-2,-3), B(4,7)$ and $C(9,-5)$ non-collinear?

Solution:

$A(-2,-3), B(4,7)$ and $C(9,-5)$ are given three points.

Now find the distances $|\overline{AB}|, |\overline{BC}|$ and $|\overline{AC}|$ by using distance formula.

$$\therefore \quad |\overline{AB}| = \sqrt{(4+2)^2+(7+3)^2} = \sqrt{36+100} = \sqrt{136} = 2\sqrt{34} \text{ units,}$$

$$\Rightarrow \quad |\overline{BC}| = \sqrt{(9-4)^2+(-5-7)^2} = \sqrt{25+144} = \sqrt{169} = 13 \text{ units,}$$

$$\Rightarrow \quad |\overline{AC}| = \sqrt{(9+2)^2+(-5+3)^2} = \sqrt{121+4} = \sqrt{125} = 5\sqrt{5} \text{ units,}$$

Since, $|\overline{AC}| \neq |\overline{AB}|+|\overline{AC}|$

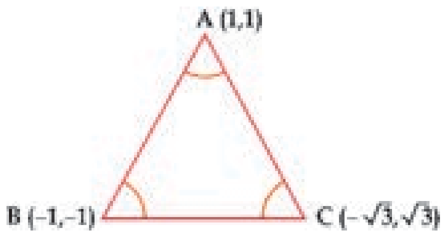
Thus, the given three points A, B and C are not collinear.

16.2.3 Use distance formula to show that the given three non-collinear points forms.

(i) An equilateral triangle,
(ii) An isosceles triangle,
(iii) A right angled triangle,
(iv) A scalene triangle.

Example 01 Show that the three points $A(1,1)$, $B(-1,-1)$ and $C(-\sqrt{3},\sqrt{3})$ form an equilateral triangle.

Solution:



$A(1,1)$, $B(-1,-1)$ and $C(-\sqrt{3},\sqrt{3})$ are given points.

Now, find the distance $|\overline{AB}|, |\overline{BC}|$ and $|\overline{AC}|$ of the sides of a ΔABC using distance formula

$$|\overline{AB}| = \sqrt{(-1-1)^2+(-1-1)^2} = \sqrt{4+4} = \sqrt{8} \text{ units,}$$

$$|\overline{BC}| = \sqrt{\left(-\sqrt{3}+1\right)^2+\left(\sqrt{3}+1\right)^2} = \sqrt{3-2\sqrt{3}+1+3+2\sqrt{3}+1} = \sqrt{8} \text{ units,}$$

and $$|\overline{AC}| = \sqrt{\left(-\sqrt{3}-1\right)^2+\left(\sqrt{3}-1\right)^2} = \sqrt{4+4} = \sqrt{8} \text{ units,}$$

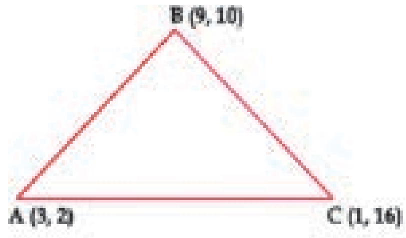
Since, $|\overline{AB}| = |\overline{BC}| = |\overline{AC}| = 2\sqrt{2}$ units, and the points are non-collinear.

Therefore ABC is an equilateral triangle.

Example 02 Show that the points $A(3,2), B(9,10)$ and $C(1,16)$ form an isosceles triangle.

Solution:

Let $A(3,2), B(9,10)$ and $C(1,16)$ are given points.



By using distance formula

$$|\overline{AB}| = \sqrt{(9-3)^2+(10-2)^2} = \sqrt{36+64} = \sqrt{100} = 10 \text{ units},$$

$$|\overline{AC}| = \sqrt{(1-3)^2+(16-2)^2} = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2} \text{ units},$$

$$|\overline{BC}| = \sqrt{(1-9)^2+(16-10)^2} = \sqrt{64+36} = \sqrt{100} = 10 \text{ units},$$

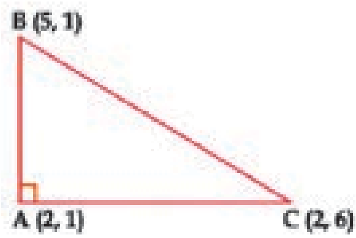
Since, $|\overline{AB}| = |\overline{BC}| = 10$ unit and points are non-collinear.

Thus, the two sides are equal in length,
Therefore ABC is an isosceles triangle.

Example 03 Show that A (2, 1), B (5, 1) and C (2, 6) are the vertices of a right angled triangle.

Solution:

Let A (2, 1), B (5, 1) and C (2, 6) be the vertices of a ΔABC.



Using distance formula,

$$|\overline{AB}|^2 = (5-2)^2+(1-1)^2 = 9+0 = 9$$

$$|\overline{BC}|^2 = (2-5)^2+(6-1)^2 = 9+25 = 34$$

and $$|\overline{AC}|^2 = (2-2)^2+(6-1)^2 = 0+25 = 25$$

Here,

$$|\overline{AB}|^2 + |\overline{AC}|^2 = 9+25 = 34$$

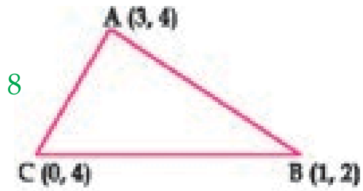
$$\Rightarrow \quad |\overline{AB}|^2 + |\overline{AC}|^2 = |\overline{BC}|^2$$

By converse of Pythagoras theorem given vertices form a right angled triangle.

Example 04 Show that the point $A(3,4), B(1,2)$ and $C(0,4)$ form a scalene triangle.

Solution:

Let $A(3,4), B(1,2)$ and $C(0,4)$ are the given points.
Now, find the length of each side using distance formula



$$\therefore \quad |\overline{AB}| = \sqrt{(1-3)^2 + (2-4)^2} = \sqrt{4+4} = \sqrt{8} \text{ units,}$$

$$|\overline{BC}| = \sqrt{(0-1)^2 + (4-2)^2} = \sqrt{1+4} = \sqrt{5} \text{ units,}$$

$$\text{and} \quad |\overline{AC}| = \sqrt{(0-3)^2 + (4-4)^2} = \sqrt{9} = 3 \text{ units,}$$

Since, $|\overline{AB}| \neq |\overline{BC}| \neq |\overline{AC}|$ and the points are non-collinear.

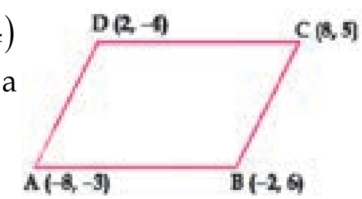
i.e. length of all the three sides are not equal.
Thus ABC is a scalene triangle.

16.2.4 Use distance formula to show that four non-collinear points form:
(i) A parallelogram.
(ii) A rectangle,
(iii) A square

Example 01 Show that $A(-8,-3), B(-2,6), C(8,5)$ and $C(2,-4)$ consecutive vertices of a parallelogram.

Solution:

Let $A(-8,-3), B(-2,6), C(8,5)$ and $C(2,-4)$ be the any four consecutive vertices of a quadrilateral ABCD.
Using Distance Formula,
We have,



$$|\overline{AB}| = \sqrt{(-2+8)^2 + (6+3)^2} = \sqrt{36+81} = \sqrt{117} \text{ units,}$$

$$|\overline{BC}| = \sqrt{(8+2)^2 + (5-6)^2} = \sqrt{100+1} = \sqrt{101} \text{ units,}$$

$$|\overline{DC}| = \sqrt{(2-8)^2 + (-4-5)^2} = \sqrt{36+81} = \sqrt{117} \text{ units,}$$

and $|\overline{AD}| = \sqrt{(2+8)^2 + (-4+3)^2} = \sqrt{100+1} = \sqrt{101}$ units,

Now, $|\overline{AB}| = |\overline{CD}| = \sqrt{117}$

and $|\overline{BC}| = |\overline{AD}| = \sqrt{101}$

$\therefore$ A, B, C and D are the vertices of parallelogram.

Example 02 Show that the four points $A(0,-1)$, $B(4,-3)$, $C(8,5)$ and $D(4,7)$ are the consecutive vertices of a rectangle.

Solution:

Let, $A(0,-1)$, $B(4,-3)$, $C(8,5)$ and $D(4,7)$ be any four consecutive points of a quadrilateral ABCD.

Using distance formula,

i.e., $d = |\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ unit

We have,

$$\therefore \quad |\overline{AB}| = \sqrt{(4-0)^2 + (-3+1)^2} = \sqrt{16+4} = \sqrt{20} \text{ units,}$$

$$|\overline{BC}| = \sqrt{(8-4)^2 + (5+3)^2} = \sqrt{16+64} = \sqrt{80} \text{ units,}$$

$$|\overline{DC}| = \sqrt{(4-8)^2 + (7-5)^2} = \sqrt{16+4} = \sqrt{20} \text{ units,}$$

$$|\overline{AD}| = \sqrt{(4-0)^2 + (7+1)^2} = \sqrt{16+64} = \sqrt{80} \text{ units,}$$

$$|\overline{AC}| = \sqrt{(8-0)^2 + (5+1)^2} = \sqrt{64+36} = 10 \text{ units,}$$

$$|\overline{BD}| = \sqrt{(4-4)^2 + (7+3)^2} = \sqrt{0+100} = 10 \text{ units,}$$

$$\therefore \quad |\overline{AB}| = |\overline{CD}| = \sqrt{20}$$

$$|\overline{BC}| = |\overline{AD}| = \sqrt{80}$$

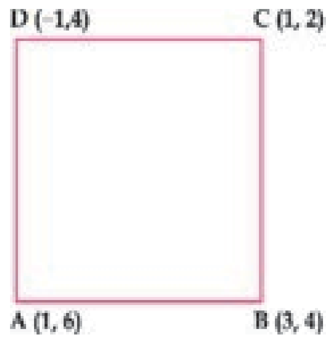
$|\overline{AC}| = |\overline{BD}|$ (Diagonals are equal)

A, B, C and D are the vertices of a rectangle.

Example 03 Show that the four consecutive points $A(1,6)$, $B(3,4)$, $C(1,2)$ and $D(-1,4)$ form a square.

Solution:

Given that $A(1,6)$, $B(3,4)$, $C(1,2)$ and $D(-1,4)$ be the four consecutive points of a quadrilateral ABCD.



By using the distance formula

$$\therefore \quad |\overline{AB}| = \sqrt{(3-1)^2+(4-6)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \text{ units,}$$

$$|\overline{BC}| = \sqrt{(1-3)^2+(2-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \text{ units,}$$

$$|\overline{DC}| = \sqrt{(1+1)^2+(2-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \text{ units,}$$

$$\text{and} \quad |\overline{AD}| = \sqrt{(-1-1)^2+(4-6)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \text{ units,}$$

Since, $|\overline{AB}| = |\overline{BC}| = |\overline{DC}| = |\overline{AD}| = 2\sqrt{2}$ unit, i.e. four sides are equal in length.

Now, find the lengths of the diagonals $\overline{AC}$ and $\overline{BC}$ respectively

$$\therefore \quad |\overline{AC}| = \sqrt{(1-1)^2+(2-6)^2} = \sqrt{0+16} = 4 \text{ units,}$$

$$\text{and} \quad |\overline{BD}| = \sqrt{(-1-3)^2+(4-4)^2} = \sqrt{16-0} = 4 \text{ units,}$$

Since, $|\overline{AC}| = |\overline{BD}| = 4$ unit, i.e. lengths of the diagonals are equal.

Hence, a quadrilateral ABCD is a Square.

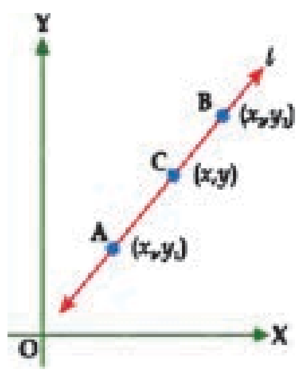
Exercise 16.2

1. Show that the points P (-3, -4), Q (2, 6) and R (0, 2) are collinear.
2. Show that the points A (-1, 0), B (1, 0) and C (0, $\sqrt{3}$) are not collinear.
3. Show that L (0, $\sqrt{3}$), M (-1, 0) and N (1, 0) form an equilateral triangle.
4. Whether or not the points A (2, 3), B (8, 11) and C (0, 17) form an isosceles triangle.
5. Do the points A (-1, 2), B (7, 5) and C (2, -6) form a right angled triangle.
6. If the points A (3, 1), B (9, 1) and C (6, k) determine an equilateral triangle, find the values of k.
7. Show that the points P (1, 2), Q (3, 4) and R (0, -1) are the vertices of a scalene triangle.
8. Show that the points A (2, 3), B (8, 11), C (0, 17) and D (-6, 9) are vertices of a square.
9. Explain why the points A (-2, 0), B (0, -3), C (2, 0) and D (0, 3) do determine a square?
10. Show that the points A (3, 2), B (4, 1), C (5, 4) and D (6, 3) are the vertices of a rectangle.
11. Use distance formula to show that the points O (0, 0), A (3, 0), B (5, 2) and C (2, 2) form the vertices of a parallelogram.

16.3 Mid-Point Formula

16.3.1 Recognize the formula to find the mid-point of the line joining two given points.

Let A (x_1, y_1) and B (x_2, y_2) be any two point of the $\overline{AB}$ in the plane and C (x, y) be the midpoint of AB, then $C(x,y)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ is called mid point of A and B.



Example 01 Find the mid-point of the line segment joining A (2, 1) and B (3, 4)

Solution:

A (2, 1) and B (3, 4) are the points of the line segment.

Mid point = ?

Using mid-point formula

$\therefore$ Mid-point of $\overline{AB} = \left(\frac{2+3}{2}, \frac{1+4}{2}\right) = \left(\frac{5}{2}, \frac{5}{2}\right)$

Example 02 If $A(2, 1)$, $B(5, 2)$, and $C(3, 4)$ are the vertices of a ΔABC, find the mid-points P, Q and R of the sides $\overline{AC}$, $\overline{AB}$ and $\overline{BC}$ respectively of a ΔABC.

Solution:

Using mid-point formula,

$C(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1+y_2}{2}\right)$, we have,

$\therefore$ Mid-point of $\overline{AC} = P = \left(\frac{2+3}{2}, \frac{1+4}{2}\right) = \left(\frac{5}{2}, \frac{5}{2}\right)$,

Mid-point of $\overline{AB} = Q = \left(\frac{2+5}{2}, \frac{1+2}{2}\right) = \left(\frac{7}{2}, \frac{3}{2}\right)$,

and, Mid-point of $\overline{BC} = R = \left(\frac{5+3}{2}, \frac{2+4}{2}\right) = (4, 3)$,

Thus, the required mid-points of the sides of a ΔABC are

$\left(\frac{5}{2}, \frac{5}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right)$ and $(4, 3)$.

16.4 Apply Distance And Midpoint Formulas To Solve/Verify Different Standards Results Related To Geometry

Example 01 Prove analytically that the length of the median to the hypotenuse of a right triangle is half the length of the hypotenuse.

Solution:

Let ABC be the triangle right-angled at B. Take B as a origin and $\overline{BC}$, $\overline{BA}$ as the axes of x and y respectively as shown in the figure

Let $|\overline{BC}| = a, |\overline{BA}| = b$ so that B is $(0,0)$, C is $(a,0)$ and A is $(0,b)$.

Therefore M, the midpoint of $\overline{AC}$ is $\left(\frac{a+0}{2}, \frac{b+0}{2}\right)$, i.e. $\left(\frac{a}{2}, \frac{b}{2}\right)$.

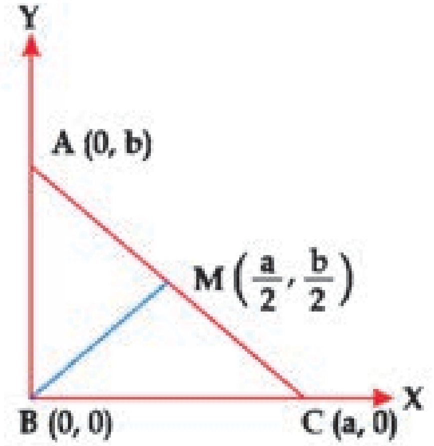
Now $|\overline{AM}| = |\overline{CM}| = \frac{1}{2}|\overline{AC}|$

$$= \frac{1}{2}\sqrt{|\overline{AB}|^2 + |\overline{BC}|^2}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$

and $|\overline{BM}| = \sqrt{\left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - 0\right)^2}$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$



Therefore

$$|\overline{BM}| = \frac{1}{2}|\overline{AC}|$$

Hence the length of median $|\overline{BM}|$ is half the length of the hypotenuse $|\overline{AC}|$.

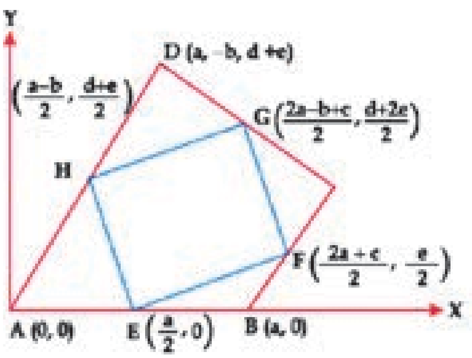
Example 02 Prove that the figure obtained by joining in order the midpoints of the sides of any quadrilateral is parallelogram.

Solution:

Let ABCD be a quadrilateral and E,F,G and H respectively be the midpoints of the sides $\overline{AB}, \overline{BC},, \overline{CD}$ and $\overline{DA}$.

Let A be taken as origin, $\overline{AB}$ as x-axis and a line through A and perpendicular to $\overline{AB}$ as y-axis

Then A is $(0,0)$ again let B be $(a,0)$,C be $(a+c, e)$ and D be $(a-b, d+e)$



Now the midpoints E,F,G and H are respectively $\left(\frac{a}{2}, 0\right), \left(\frac{2a+c}{2}, \frac{e}{2}\right), \left(\frac{2a-b+c}{2}, \frac{d+2e}{2}\right)$ and $\left(\frac{a-b}{2}, \frac{d+e}{2}\right)$.

Therefore, by the distance formula

$$|\overline{EF}|^2 = \left(\frac{2a+c}{2} - \frac{a}{2}\right)^2 + \left(\frac{e}{2} - 0\right)^2$$

$$= \left(\frac{a+c}{2}\right)^2 + \left(\frac{e}{2}\right)^2$$

$$= \frac{1}{4}\left\{(a+c)^2 + e^2\right\} \qquad \text{(i)}$$

$$|\overline{GH}|^2 = \left(\frac{2a-b+c}{2} - \frac{a-b}{2}\right)^2 + \left(\frac{d+2e}{2} - \frac{d+e}{2}\right)^2$$

$$= \left(\frac{a+c}{2}\right)^2 + \left(\frac{e}{2}\right)^2$$

$$= \frac{1}{4}\left\{(a+c)^2 + e^2\right\} \qquad \text{(ii)}$$

From (i) and (ii) , we have

$$|\overline{EF}| = |\overline{GH}|$$

Similarly, $|\overline{GF}| = |\overline{EH}|$

Since the opposite sides are equal, EFGH is a parallelogram.

Exercise 16.3

1. Find the mid-points between the following pair of points using mid-point formula.
   (i) A (2, 6) and B (-4, 8) (ii) P (-3, -1) and Q (5, 2)
   (iii) L (0, 6) and M (-8,0) (iv) C (0, 0) and D $(2\sqrt{3}, 4\sqrt{3})$.
2. Find the centre of a circle whose end points of a diameter are A (-5, 6) and B (3, -4).
3. The centre of a circle is (3, 4) and one of its end point of a diameter is (4,6), find the point of other end.
4. A circle has a diameter between the points A (-3, 4) and B (11, 6). Find the centre and radius of the circle.
5. Prove that a triangle is an isosceles triangle if and only if it has two equal medians.
6. Prove that the diagonals of a parallelogram bisect each other.

Review Exercise 16

1. Read the following sentences carefully and encircle "T" in case of True and "F" in case of False statement.

(i) R is the mid-poin of $\overline{PQ}$, if R is lying between P and Q T/F
(ii) In scalene triangle all sides are equal T/F
(iii) Perpendicular lines meet at an angle of 135°. T/F
(iv) Collinear points may form a triangle. T/F
(v) Non-collinear points form a triangle. T/F
(vi) In an isosceles triangle, two sides angles are equal. T/F
(vii) All the points that lie on the y-axis are collinear T/F
(viii) Intersection of x-axis and y-axis is (0,0) T/F
(ix) The distance from origin to (6,0) is 36 unit. T/F

2. Fill in the blanks.

(i) If $A(x_1, y_1)$ and $B(x_2, y_2)$ be the any two points on the line, then $|\overline{AB}|=$ __________
(ii) Collinear points lies on the same __________
(iii) In 4th quadrant $x>0$ and y__________

3. Tick (✓) the correct answer.

(i) Two perpendicular lines meet at an angle of:
(a) 45° (b) 60°
(c) 90° (d) 180°

(ii) $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$ is called:
(a) Mid-point formula (b) Distance formula
(c) Division formula (d) Ratio formula

(iii) $A(3, 0)$ and $B(0, 3)$ are any two points in the plane then $|\overline{AB}|=$
(a) 6 unit (b) $6\sqrt{2}$ unit
(c) $3\sqrt{2}$ unit (d) $3\sqrt{2}$ unit

(iv) Three points A,B and C are collinear if
(a) $m\,\overline{AB} = m\,\overline{BC} + m\,\overline{AC}$ (b) $(m\text{ AB})^2 = (m\text{ BC})^2 + (m\text{ AC})^2$
(c) both a and b (d) $(m\,\overline{AB}) \neq m\,\overline{BC} + m\,\overline{AC}$

Summary

- The distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by $d = |\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ units.
- The coordinates of the mid-point of line segment AB, passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $M(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- Collinear points form a straight line.
- For collinearity of three points *A*, *B* and *C*,

  either $|\overline{AC}| = |\overline{AB}| + |\overline{BC}|$,

  or $|\overline{AB}| = |\overline{AC}| + |\overline{CB}|$

  or $|\overline{BC}| = |\overline{BA}| + |\overline{AC}|$ holds good
- Three non-collinear points *A*, *B* and *C* form a triangle, if the sum of the lengths of any two sides is greater than the length of the third side.
- If $|\overline{AB}| = |\overline{BC}| < |\overline{AC}|$, then no triangle can be formed by the points *A*, *B* and *C*.

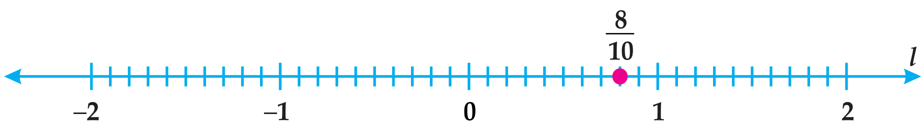
Answers

Exercise 1.1

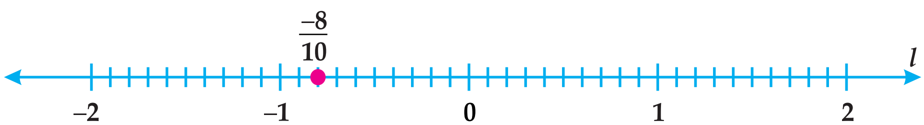
1. i. Rational number | ii. Irrational number
iii. Irrational number | iv. Rational number
v. Irrational number | vi. Irrational number
vii. Rational number | viii. Irrational number
ix. Irrational number | x. Rational number
xi. Irrational number | xii. Rational number

2. i. Terminating | ii. Non-terminating
iii. Non-terminating | iv. Terminating
v. Terminating | vi. Non-terminating

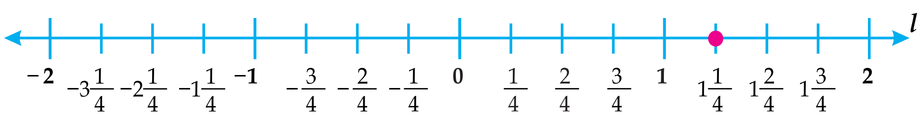
3. i.



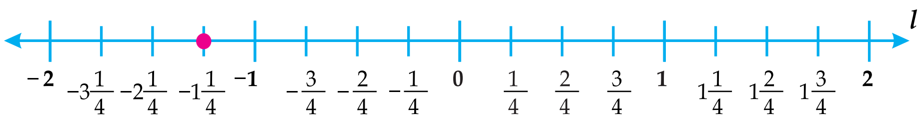
ii.



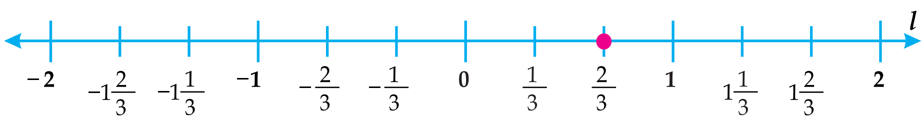
iii.



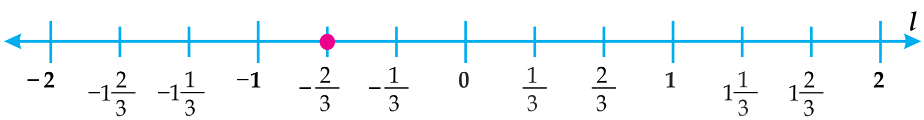
iv.



v.



vi.



4. We can not make list of all real numbers between 1 and 2.

5. Pi (π) is an irrational number because it is non terminating and non recurring decimal.

6. **i.** False **ii.** True **iii.** True **iv.** True **v.** True **vi.** False

Exercise 1.2

1.
- **i.** Commutative property of addition.
- **ii.** Associative property of addition.
- **iii.** Left distributive property of multiplication over addition.
- **iv.** Right distributive property of multiplication over addition.
- **v.** Right distributive property of multiplication over subtraction.
- **vi.** Commutative property of multiplication.
- **vii.** Associative property of multiplication.
- **viii.** Multiplicative inverse.
- **ix.** Additive inverse.
- **x.** Multiplicative inverse.
- **xi.** Left distributive property of multiplication over subtraction.
- **xii.** Multiplicative inverse.

2. **i.** $\frac{\sqrt{2}}{5}+\frac{3}{\sqrt{6}}=\frac{\boxed{3}}{\sqrt{6}}+\frac{\sqrt{2}}{\boxed{5}}$ **ii.** $\frac{7}{10}+\left(\frac{70}{\boxed{10}}+\frac{16}{33}\right)=\left(\frac{7}{\boxed{10}}+\frac{\boxed{70}}{10}\right)+\frac{16}{\boxed{33}}$

iii. $\frac{99}{50}\times\frac{50}{99}=\boxed{1}$ **iv.** $\frac{59}{95}\times\frac{95}{59}=\boxed{1}$

v. $(-21)+\left(\boxed{21}\right)=0$ **vi.** $\frac{5}{8}\times\left(\frac{2}{3}+\frac{5}{7}\right)=\left(\frac{\boxed{5}}{\boxed{8}}\times\frac{2}{3}\right)+\left(\frac{5}{8}\times\frac{\boxed{5}}{\boxed{7}}\right)$

3. **i.** $5<10$ **ii.** $10>5$ **iii.** $6+9$
iv. $6+8$ **v.** $6+6$

4. **i.** 7×12 **ii.** 5×12 **iii.** $<$ **iv.** $>$

5. Additive inverse | multiplicative inverse.

	Additive inverse	multiplicative inverse.
i.	-3	$\frac{1}{3}$
ii.	7	$\frac{-1}{7}$
iii.	-0.3	$\frac{1}{0.3}$
iv.	$\frac{\sqrt{5}}{5}$	$\frac{-5}{\sqrt{5}}$
v.	$\frac{-9}{\sqrt{12}}$	$\frac{\sqrt{12}}{9}$
vi.	0	does not exist

Exercise 1.3

1. i. Radicand $= 5$, Index $= 3$

ii. Radicand $= \frac{x}{y}$, Index $= 4$

iii. Radicand $= x^2yz$, Index $= 5$

iv. Radicand = ab, index =2

v. Radicand = $\frac{pq}{r}$, index =n

2. i. $\left(\frac{3}{4}\right)^{\frac{1}{2}}$ ii. $\left(\frac{x}{y}\right)^{\frac{5}{2}}$ iii. $\left(\frac{x}{y}\right)^{\frac{5}{3}}$ iv. $(yz)^{\frac{7}{3}}$ v. $(27)^{\frac{1}{9}}$

vi. $(-64)^{\frac{2}{3}}$ vii. $\left(\frac{1}{2}\right)^{\frac{m}{3}}$ viii. $(xy)^{\frac{3}{5}}$ ix. $\left(\frac{4}{3}\right)^{\frac{1}{6}}$

3. i. $\sqrt[7]{(5)^3}$ ii. $\sqrt[3]{\frac{a}{b^2}}$ iii. $\sqrt[7]{\left(\frac{5}{7}\right)^{15}}$ iv. $\sqrt{\left(\frac{a}{b}\right)^m}$ v. $\sqrt[5]{\left(\frac{11}{13}\right)\left(\frac{12}{13}\right)}$

Exercise 1.4

1. i. 27 ii. 20 iii. $(a+b)(c+d)$

2. i. $\left(\frac{1}{3}\right)^9$ ii. $\left(\frac{3}{4}\right)^7$ iii. $\left(\frac{4}{5}\right)^8$ iv. $-3^3\times5^6$ v. $3^3\times4^6$

vi. $\frac{a^9}{b^9c^9}$ vii. $\frac{c^{10}}{d^{10}}$ viii. $m^6n^5t^{11}$ ix. $a^9b^6c^8$

3. i. 5^6 ii. $x^{15}y^{15}$ iii. $(4)^{10}$ iv. $-3^9\times4^6$ v. $\frac{b^6}{5^3}$

vi. $\frac{(4)^6}{9^6}$ vii. z^{24} viii. m^{100} ix. $-(0.1)^{18}$

Exercise 1.5

1. i. $1+2i$ ii. $2+2i$ iii. $4i$
iv. $-1+i$ v. -2 vi. $-3+4i$

2. i. $\text{Re}(z)=1, \text{Im}(z)=2$ ii. $\text{Re}(z)=4, \text{Im}(z)=9$

iii. $\text{Re}(z)=-5, \text{Im}(z)=6$ iv. $\text{Re}(z)=-1, \text{Im}(z)=-1$

v. $\text{Re}(z)=\frac{-3}{4}, \text{Im}(z)=\frac{4}{5}$ vi. $\text{Re}(z)=-1, \text{Im}(z)=2$

3. i. $\bar{z}=3-2i$ ii. $\bar{z}=(0, 7)$ iii. $\bar{z}=(-1, 0)$

iv. $\bar{z}=1+i$ v. $\bar{z}=\frac{-3}{4}+\frac{4}{5}i$ vi. $\bar{z}=1-3i$

5. i. $x=-5, y=5$ ii. $x=\pm\frac{4}{3}$, $y=\pm\frac{3}{5}$

iii. $x=\frac{-27}{5}, y=\pm 11$ iv. $x=\frac{9\sqrt{30}}{\sqrt{5}}, y=\frac{-4}{27}$

Exercise 1.6

1. i. $(12,5)$ ii. $\left(\frac{13}{6},\frac{13}{6}\right)$ iii. $(5,21)$

iv. $\left(0,-\frac{1}{15}\right)$ v. $(5,0)$ vi. $(0,-41)$

vii. $\left(\frac{3-6\sqrt{2}}{4},\frac{3+3\sqrt{2}}{2\sqrt{2}}\right)$ viii. $\left(\frac{-5}{13},\frac{-27}{13}\right)$

2. i. $\frac{-1}{2}+\frac{1}{2}i$ ii. -4 iii. $-\frac{i}{2}$ iv. 16

Review Exercise 1

1. i. $\frac{1}{\sqrt{5}}$ ii. Set of real numbers iii. 0
iv. 7 v. 0 vi. Irrational
vii. rational viii. $-3-5i$ ix. 2
x. (ac – bd, ad + bc)

2. i. True ii. True iii. True iv. True v. True

3. i. a ii. b iii. c iv. b

4. i. 2 ii. $\frac{1}{3}$ 5. i. 3^{21} ii. 2^{36}

6. i. 7 ii. -1 iii. $7+i$ iv. $5\sqrt{2}$ v. $\frac{7}{50}+\frac{1}{50}i$
vi. $\frac{1}{5\sqrt{2}}$ vii. $1+7i$ viii. $-1+7i$

Exercise 2.1

1. i. 9.7×10^{3} ii. 4.98×10^{6} iii. 9.6×10^{7} iv. 4.169×10^{3}
v. 8.4×10^{4} vi. $7.18\text{x}10^{-1}$ vii. $6.43\text{x}10^{-3}$ viii. $7.4\text{x}10^{-3}$
ix. $2.1005\text{x}10^{-1}$

2. i. 70000 ii. 0.0000000008072 iii. 6018000 iv. 786500000
v. 0.000205 vi. 72500000000 vii. 4502000 viii. 0.00000002865
ix. 3056000

Exercise 2.2

1. i. $\log_7 343 = 3$ ii. $\log_3 \frac{1}{81} = -4$ iii. $\log_{10}(0.001) = -3$
iv. $\log_8 (4) = \frac{2}{3}$

2. i. $(27)^{\frac{4}{3}} = 81$ ii. $(2)^{-3} = \frac{1}{8}$ iii. $10^{\circ} = 1$ iv. $(10)^{-2} = 0.01$

3. i. $x = 4\sqrt{2}$ ii. $a = 9$ iii. $y = 4$ iv. $x = 8$
v. $y = 2$ vi. $a = 4$ vii. a is any positive real number
viii. $y = 1$ ix. $x = 1$

Exercise 2.3

1. i. Characteristic : 0, Mantissa : 0.9031
   ii. Characteristic : 3, Mantissa : 0.7036
   iii. Characteristic : 0, Mantissa : 0.9997
   iv. Characteristic : 2, Mantissa : 0.8839
   v. Characteristic : –3, Mantissa : 0.5172
   vi. Characteristic : –5, Mantissa : 0.4771
2. i. 0.9542 ii. 1.7448 iii. 1.4711 iv. 2.6078
   v. $\overline{3}.6712$ vi. $\overline{5}.8808$
3. i. 0.4926 ii. 2.4926 iii. $\overline{3}.4926$ iv. 3.4926
   v. 2.4926 vi. 5.4926

Exercise 2.4

1. i. 3692 ii. 0.5530 iii. 2.278
   iv. 653800 v. 0.0002425 vi. 8.292
2. i. 2.954242509 ii. 1.658393026 iii. 4.563267445
   iv. 2.917137753 v. –2.07007044 vi. – 4.013228266
3. i. 56.2989 ii. 4.5803 iii. 0.024367
   iv. 3019.95 v. 0.0000000991 vi. 1.8471

Exercise 2.5

1. i. $\log_a x+\log_a y+\log_a z$ ii. $2\log_a x+\log_a y$
   iii. $\log_a x+\log_a y-\log_a z$ iv. $\frac{1}{2}\log_a x+\frac{1}{2}\log_a y$
   v. $-\frac{1}{2}\log_a x-\frac{1}{2}\log_a y-\frac{1}{2}\log_a z$ vi. $3\log_a x+\log_a y-2\log_a z$
   vii. $\frac{1}{2}\log_a x+\log_a y+\frac{1}{2}\log_a z$ viii. $\frac{-7}{12}\log_a(x)-\log_a y$
   ix. $\frac{-2}{3}\log_a x+\frac{3}{2}\log_a y-\frac{2}{3}\log_a z$
2. i. $\log_a(2\sqrt{2})$ ii. $\log_a\left(x^2-1\right)$ iii. $\log\frac{(x+1)^2}{x(x-1)}$

3. i. 1.1761 ii. 1.8062 iii. 0.5 iv. 1.6812
v. 0.6276 vi. 1.4771 vii. 0.4260 viii. 0.4604

Exercise 2.6

1. i. 253.688 ii. 6750 iii. 48.2176 iv. $x = 930.80$
v. 15.20 vi. 1.2585 vii. 410130 viii. 1.84077×10^{13}

2. i. 8 ii. 22 iii. 15 iv. 14 v. 29

Review Exercise 2

1. i. False ii. False iii. True iv. False v. True
2. i. Common logarithm ii. 0 iii. Mantissa
iv. 9 v. $\log_a n$ vi. $a^x = y$
vii. $\log_a y = 10$ viii. 1 ix. $\log_a m - \log_a n$
x. 2 xi. 0 xii. −5
3. i. *d* ii. *b* iii. *b* iv. *b* v. *a*
vi. *c* vii. *c* viii. *b* ix. *b* x. *c*

Exercise 3.1

1. i. Polynomial ii. Not a polynomial iii. Polynomial
iv. Not a polynomial v. Not a polynomial vi. Not a polynomial
2. i. Rational ii. Not a rational iii. Rational
iv. Not a rational v. Rational vi. Not a rational
3. i. p−10 ii. $\dfrac{a}{a+b}$ iii. $\dfrac{a}{2(a+b)}$
iv. $x+y-z$ v. $\dfrac{3m(m+5)}{2}$ vi. $\dfrac{x-3}{x-2}$

4. i. $\frac{4x^2-1}{x^2-1}$ ii. $\frac{3x+7}{(x+2)(x+3)}$ iii. $\frac{2\ x^2y^2+xy+1}{(xy+1)(xy-1)}$

iv. $\frac{-15}{(x+3)\ (x+6)}$ v. $\frac{-2b}{a^2-b^2}$ vi. $\frac{-(y-1)}{y+1}$

5. i. $\frac{8y^3}{(2y-x)^2\ (2y+x)}$ ii. $-\frac{2x+3y}{y}$ iii. 1

iv. $\frac{5}{3}$ v. $\frac{(q-5)(q+3)}{q^2}$ vi. $\frac{8(z-1)}{z-5}$

6. $\frac{2(x^2+y^2)}{x^2-y^2}$

7. i. $\frac{1}{6}$ ii. $9\frac{9}{55}$ iii. $-\frac{17}{73}$

iv. $-4\frac{4}{9}$ v. $1\frac{1}{11}$

Exercise 3.2

1. $a^2+b^2=50,\ \ ab=7$ **2.** $a^2+b^2=17,\ \ ab=4$ **3.** $a^2+b^2+c^2=55$

4. $a^2+b^2+c^2=\frac{5}{9}$ **5.** $a+b+c=\pm 7$ **6.** $a+b+c=\pm\sqrt{2.5}$

7. $ab+bc+ca=40$ **8.** $a^3+b^3=28$ **9.** $ab=-8$

10. $ab=-4$ **11.** $a^3-b^3=230$ **12.** $125x^3+y^3=247$

13. $216a^3-343\ b^3=12419$ **14.** $x^3+\frac{1}{x^3}=322$ **15.** $x^3-\frac{1}{x^3}=1364$

16. i. $\frac{27b^3}{8}+\frac{8}{27b^3}$ ii. $\frac{343y^6}{729}+\frac{729}{343y^6}$

iii. $\frac{x^{12}}{1728}-\frac{1728}{x^{12}}$ iv. $c^6-\frac{1}{c^6}$

17. i. $8x^6+27y^6$ ii. $8x^6-27y^6$

iii. $x^{12}-y^{12}$ iv. $256x^8-6561y^8$

Exercise 3.3

1. i. $\frac{3z}{x^2}$ ii. $4\sqrt[3]{4a^2b^4c^3}$ iii. 2 iv. $36\sqrt{6}$

v. $\frac{25}{32}$ vi. $\frac{14\sqrt{3}}{11}$ vii. 6 viii. 2

2. i. $8+4\sqrt{3}$ ii. $6\sqrt{6}-2\sqrt{3}$ iii. $8\sqrt{12}-\sqrt{8}$ iv. $2+\sqrt{3}$

3. i. $66\sqrt{2}$ ii. $13\sqrt{5}$ iii. $20+9\sqrt{3}$ iv. $15\sqrt{10}$

v. $4\sqrt{5}+25$ vi. $\sqrt{11}$ vii. 136 viii. $3\sqrt{2}$

ix. $\frac{1}{3}$ x. 2 xi. $134-24\sqrt{30}$ xii. $30+12\sqrt{6}$

Exercise 3.4

1. i. $2-\sqrt{3}$ ii. $3-2\sqrt{2}$ iii. $-\left(\frac{5\sqrt{2}+4\sqrt{3}}{2}\right)$

iv. $16\left(2\sqrt{3}-11\right)$ v. $\frac{83-18\sqrt{2}}{79}$ vi. $\frac{11+3\sqrt{3}}{2}$

2. i. $\left(x+\frac{1}{x}\right)^2=256$ ii. $x=-\left(\frac{4\sqrt{7}+11}{9}\right)$

iii. $x+\frac{1}{x}=6, x-\frac{1}{x}=-4\sqrt{2}, x^2+\frac{1}{x^2}=34, x^2-\frac{1}{x^2}=-24\sqrt{2}, x^4+\frac{1}{x^4}=1154$

3. 322 **4.** 194 **5.** $112\sqrt{3}$

Review Exercise 3

1.

i.	b	ii.	b	iii.	a
iv.	a	v.	a	vi.	a
vii.	b	viii.	a	ix.	a
x.	a	xi.	a	xii.	b

2. i. Highest power or exponent on the variable

ii. $2+\sqrt{3}$ iii. 4

iv. Irrational expression v. x^4-y^4

Exercise 4.1

1. i. $4(x+4y+6z)$ ii. $x^2(1+3y+4y^2z)$

iii. $3pq(r+2t+s)$ iv. $9qr(s^2+t^2)(1+2qr)$

v. $\frac{xz^2}{4}\left(\frac{1}{4}-\frac{x}{2}+\frac{xz}{3}\right)$ vi. $a(x-y)(1-ab+ab^2)$

2. i. $(7+z)(x+z)$ ii. $3(3ab-2c)(a+2b)$

iii. $2(t-2p)(3+2q)$ iv. $(r+9s)(r-7s)$

v. $(1-z)\left(\frac{y^2}{4}-\frac{z^2t}{9}\right)$ vi. $\frac{1}{11}(2y+z)(5x-7y)$

3. i. $(2a+3b)^2$ ii. $(6x^2+1)^2$

iii. $\left(x+\frac{1}{2x}\right)^2$ iv. $(9y+8z)^2$

v. $(25+a^2b)^2$ vi. $(a+0.2)^2$

4. i. $(b^2-2c^2)^2$ ii. $\left(\frac{3x^2}{2}-\frac{2}{3x^2}\right)^2$

iii. $2ab^3(a-4b)^2$ iv. $(3p+3q-r^2)^2$

v. $(xy-0.05)^2$ vi. $(a-b-9)^2$

5. i. $(2a-3b)(2a+3b)$ ii. $(4x-5y)(4x+5y)$

iii. $(10xz+y^2)(10xz-y^2)$ iv. $\left(\frac{x^2}{10}+10y^2\right)\left(\frac{x^2}{10}-10y^2\right)$

v. $\left(\frac{8f}{9}-\frac{9g^2}{8}\right)\left(\frac{8f}{9}+\frac{9g^2}{8}\right)$ vi. $\left(\frac{x^2}{11}-11y\right)\left(\frac{x^2}{11}+11y\right)$

6. i. $8xz$

ii. $4(3a-2b)(a-7b)$

iii. $(13x^2-3t-4)(13x^2+3t+4)$

iv. $(13x^2-5y^2)(5x^2-3y^2)$

v. $\left(a+\frac{1}{a}+b-\frac{1}{b}\right)\left(a+\frac{1}{a}-b+\frac{1}{b}\right)$

vi. $\left(3x+\frac{1}{3x}+2y-\frac{1}{2y}\right)\left(3x+\frac{1}{3x}-2y+\frac{1}{2y}\right)$

7. i. $(x+y-3z^2)(x+y+3z^2)$ ii. $(2a+2b^2-3c)(2a+2b^2+3c)$

iii. $(4d^2-c^2+d)(4d^2+c^2-d)$ iv. $(2x+2y^2+3y^3)(2x+2y^2-3y^3)$

v. $(x+y-1)(x-y-z)$ vi. $(2x+y+1)(2x-y-1)$

8. i. $\left(\sqrt{ab}-\sqrt{c}\right)\left(\sqrt{ab}+\sqrt{c}\right)$ ii. $\left(2\sqrt{x}-3\sqrt{y}\right)\left(2\sqrt{x}+3\sqrt{y}\right)$

iii. $\left(\sqrt{yz}-\frac{1}{\sqrt{yz}}\right)\left(\sqrt{yz}+\frac{1}{\sqrt{yz}}\right)$ iv. $\left(\sqrt{xzt}-\frac{1}{\sqrt{t}}\right)\left(\sqrt{xzt}+\frac{1}{\sqrt{t}}\right)$

Exercise 4.2

1. i. $\left(a^2+x^2+ax\right)\left(a^2+x^2-ax\right)$ ii. $\left(b^2-b+1\right)\left(b^2+b+1\right)$

iii. $(a^2+x^2-ax)(a^2+x^2+ax)(a^4+x^4-a^2x^2)$

iv. $\left(z^2+z+1\right)\left(z^2-z+1\right)\left(z^4-z^2+1\right)$

2. i. $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$ ii. $9\left(2x^2z^2+2xyz+y^2\right)\left(2x^2z^2-2xyz+y^2\right)$

iii. $\left(2t^2+10t+25\right)\left(2t^2-10t+25\right)$ iv. $\left(2t^2+2t+1\right)\left(2t^2-2t+1\right)$

3. i. $(x-2)(x+5)$ ii. $(ab+2)(ab-5)$

iii. $(y-7)(y+14)$ iv. $(xyz-4)(xyz+6)$

4. i. $(3y+8z)(3y-z)$ ii. $2(7x+1)(3x-1)$

iii. $(2x+1)(2x+5)$ iv. $(3x+y)(x-13y)$

Exercise 4.3

1. i. $(x-2)^2(x-7)(x+3)$ ii. $\left(x^2+5x+3\right)\left(x^2+5x+7\right)$

iii. $(x-3)(x+1)\left(x^2-2x+10\right)$ iv. $\left(x^2-8x+1\right)\left(x^2-8x-1\right)$

v. $\left(x^2+9x-2\right)\left(x^2+9x+6\right)$ vi. $(x-6)(x+1)\left(x^2-5x+16\right)$

2. i. $\left(x^2+5x-2\right)\left(x^2+5x+12\right)$ ii. $\left(x^2+7x+16\right)\left(x+6\right)\left(x+1\right)$

iii. $\left(x^2-5x+15\right)\left(x^2-5x-5\right)$ iv. $\left(x^2-12x+30\right)\left(x-8\right)\left(x-4\right)$

v. $\left(x^2-5x-10\right)\left(x^2-5x+20\right)$ vi. $\left(x^2-7x+27\right)\left(x^2-7x-5\right)$

3. i. $\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right)\left(x+2\sqrt{3}\right)\left(x-2\sqrt{3}\right)$

ii. $\left(x^2+1\right)\left(x+\sqrt{14}\right)\left(x-\sqrt{14}\right)$ iii. $\left(x+\sqrt{3}\right)^2\left(x-\sqrt{3}\right)^2$

iv. $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)\left(x+4\sqrt{2}\right)\left(x-4\sqrt{2}\right)$

v. $\left(x+\sqrt{5}\right)\left(x-\sqrt{5}\right)\left(x+2\sqrt{5}\right)\left(x-2\sqrt{5}\right)$ vi. $\left(x+2\sqrt{3}\right)\left(x-2\sqrt{3}\right)\left(x^2-2x-12\right)$

Exercise 4.4

1. i. $(b+c)^3$ ii. $(2x+y)^3$ iii. $\left(4x+\frac{1}{4}\right)^3$

iv. $(2x+3)^3$ v. $\left(\frac{1}{3}+y^2\right)^3$ vi. $\left(\frac{2}{3}x+\frac{3}{2}y\right)^3$

vii. $\left(\frac{4}{3}+x\right)^3$ viii. $\left(\frac{z}{2}+\frac{y}{3}\right)^3$

2. i. $(d-2c)^3$ ii. $\left(x^2-\frac{4}{3}\right)^3$ iii. $\left(\frac{x}{5}-y\right)^3$

iv. $\left(5z-y^2\right)^3$ v. $\left(\frac{z}{3}-6y\right)^3$ vi. $\left(\frac{b^2}{3}-\frac{c^2}{2}\right)^3$

vii. $\left(6-\frac{z}{2}\right)^3$ viii. $\left(\frac{2}{3}x-\frac{3}{2}y\right)^3$

Exercise 4.5

1. i. $\left(x+2y\right)\left(x^2-2xy+4y^2\right)$ ii. $a^2\left(a+b\right)\left(a^2-ab+b^2\right)\left(a^6-a^3b^3+b^6\right)$

iii. $\left(a^2+1\right)\left(a^4-a^2+1\right)$ iv. $\left(ab+8\right)\left(a^2b^2-8ab+64\right)$

v. $b^3(a+3b)(a^2-3ab+9b^2)$ vi. $\left(\frac{x}{5}+\frac{5}{x}\right)\left(\frac{x^2}{25}-1+\frac{25}{x^2}\right)$

vii. $x^3\,(x^2+y^2z^3)(x^4-x^2y^2z^3+y^4z^6)$

viii. $\left(\frac{x^2}{3}+\frac{2}{x}\right)\left(\frac{x^4}{9}-\frac{2x}{3}+\frac{4}{x^2}\right)$

2. i. $(x-2y)(x^2+2xy+4y^2)$ ii. $(x^3-2y^3)(x^6+2x^3y^3+4y^6)$

iii. $\left(10-\frac{xy}{5}\right)\left(100+\ xy+\frac{x^2y^2}{25}\right)$

iv. $(a+b)(a^2-ab+b^2)(a-b)(a^2+ab+b^2)$

v. $\left(\frac{x}{2}+\frac{2}{x^2}\right)\left(\frac{x^2}{4}-\frac{1}{x}+\frac{4}{x^4}\right)\left(\frac{x}{2}-\frac{2}{x^2}\right)\left(\frac{x^2}{4}+\frac{1}{x}+\frac{4}{x^4}\right)$

vi. $(x-y)(x+y)(x^2+y^2)(x^4+y^4-x^2y^2)(x^2+y^2+xy)(x^2+y^2-xy)$

vii. $\left(\frac{3}{x}-2y^2\right)\left(\frac{9}{x^2}+\frac{6y^2}{x}+\ y^4\right)$ viii. $\left(2x^2-\frac{1}{9}\right)\left(4x^4+\frac{2x^2}{9}+\frac{1}{81}\right)$

Exercise 4.6

1. i. $R=-2$ ii. $R=2$ iii. $R=18$ iv. $R=-42$ v. $R=-11$

vi. $R=\frac{3}{2}$ vii. $R=-8$ viii. $R=3y^4$

2. $m=-1$ 3. $k=-24$ 4. $r=-1,\ r=3$

Exercise 4.7

1. i. $R=0,\ q(x)=x^2+1$ ii. $R=-2,\ q(x)=x^2-2x+1$

iii. $R=-60,\ q(x)=x^2-8x+27$ iv. $R=4,\ q(x)=x^2+8x+5$

v. $R=29,\ q(x)=x^3-3x^2+7x-15$ vi. $R=1,\ q(x)=x^3+2x^2+x+2$

vii. $R=-291,\ q(x)=x^4-3x^3+10x^2-32x+96$

viii. $R=174,\ q(x)=x^4+2x^3+7x^2+18x+60$

ix. $R=175,\ q(x)=2x^3+2x^2+104x+40$

x. $R = -300,\ q(x) = 6x^3 - 42x^2 + 90x - 114$

2. $k = 24$ 3. $m = 4$ 4. $m = -24$ 5. $m = -1$

Exercise 4.8

1. i. $(x-1)(x^2+1)$ ii. $(x+1)^2(x-1)$
iii. $(x-1)(x-2)(x-3)$ iv. $(x+2)(x-2)(x+5)$
v. $(x-2)(x^2+9)$ vi. $(x+1)(2x-1)(3x+2)$
vii. $(x+1)(x+3)(x+4)$ viii. $(x+1)(2x+1)(x+3)$
ix. $(x+2)(x+4)(x+6)$

Review Exercise 4

1. i. True ii. True iii. True iv. False
v. False vi. False
2. i. $4x + y^2$ ii. $x^2 + 4xy + 16y^2$ iii. $x + 3$
iv. $2xy$ v. $a^2 - 3ab + 9b^2$
3. i. b ii. c iii. d iv. a v. b vi. c

Exercise 5.1

1. i. HCF = $24x^3y^5z^2$ ii. HCF = $6r^3s^3t^3$
iii. HCF = $(x+3)$ iv. HCF = $(2x-3)$
v. HCF = $2(a-2b)$ vi. HCF = $(x+1)$
2. i. HCF = $(x+1)$ ii. HCF = $(x^2 + 7x + 12)$
iii. HCF = $(x-2)$ iv. $x^2 + 3x + 1$
3. i. LCM = $81a^4b^5c^8$ ii. $600p^5q^4r^8$
iii. $7x(x-1)(3x-2)$ iv. $(x+4)(x+7)(x-3)$
v. $(3x+1)(x-1)(2x+3)$

vi. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

4. i. $(x-20)(x-5)(x+4)$ ii. $(3x+2)(x+4)(2x-1)$

iii. $(x+y+z)(x-y-z)(y-z-x)$ iv. $12x^3(x-4)(x+7)(x-2)$

5. $(x-8)(x^2-6x+6)$

6. x^2+2x-3

7. $9x^4+15x^3-12x-14x^2+8$

9. 6cm

10. 11 : 54 am

Exercise 5.2

1. i. $\dfrac{7x+3}{(x+1)^2}$ ii. $\dfrac{12x^2+29x+16}{3x(2x+1)(x+1)}$

iii. $\dfrac{-4x^3-x^2-9x+4}{(x+1)^2(x-3)}$ iv. $\dfrac{2(x^2+3x+3)}{(x+1)(x+2)(x+3)}$

v. $2x+6$ vi. $\dfrac{x+3}{x+9}$

vii. $\dfrac{-(x+3)(4x+7)}{(x+1)^2(x+2)}$ viii. $\dfrac{-2x+7}{(x-2)(x-3)}$

ix. $\dfrac{x^2-5x-42}{(x^2-9)(x+4)(x+5)}$ x. $\dfrac{2(x^2+5)}{4x^2+x+2}$

Exercise 5.3

1. i. $6x-5y$ ii. $3x+\dfrac{1}{x}$ iii. $2x^2y^2-\dfrac{3xy}{z^2}$

iv. $18-12x-4y$ v. $(x+\dfrac{1}{x}+1)$ vi. $3(2x-1)(x-3)$

vii. $(x-1)(x-3)$ viii. $(x+3)(x+5)(x+2)$

2. i. x^2+x+1 ii. $5x^2+4x+1$ iii. $2x^2+2x+4$

iv. $\left(\frac{x}{y}+7-\frac{y}{x}\right)$ v. $x-1+\frac{1}{x}$ vi. $x+\frac{y}{3}+3z$

vii. $x^2-4+\frac{1}{x^2}$ viii. $x^3-2+\frac{1}{x^3}$

3. 7 4. $-24x^2+9$ 5. $m=20$ 6. $p=56,\ q=49$ 7. $a=12,\ b=9$

Review Exercise 5

1. i. True ii. False iii. True iv. False v. True
2. i. two ii. $p(x)\ q(x)$ iii. 1
   iv. $(y+1)(y+2)(y+3)$ v. $y+\frac{1}{y}$
3. i. d ii. d iii. b iv. c v. b vi. d
   vii. c viii. b ix. b x. c

Exercise 6.1

1. i. $x=20$ ii. $x=-12$ iii. $x=30$ iv. $x=40$
   v. $y=\frac{1}{15}$ vi. $y=\frac{11}{20}$ vii. $x=\frac{44}{17}$ viii. $x=105$
   ix. $\frac{105}{13}$ x. $x=1$ xi. $x=-\frac{20}{7}$ xii. $x=12$
   xiii. $x=-\frac{5}{4}$ xiv. $x=-4$ xv. $m=\frac{-11}{6}$
2. $\{5\}$ 3. $x=7$ 4. Bilal = 18 years old
   Ali = 12 years old
5. i. $\{1\}$ ii. $\{10\}$ iii. $\{100\}$
   iv. $\{-12\}$ v. $\{143\}$ vi. $\{\ \}$
   vii. $\{2\}$ viii. $\{81\}$ ix. $\{80\}$

Exercise 6.2

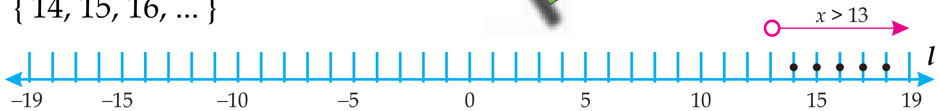
1. i. $\left\{-\frac{7}{2}, \frac{5}{2}\right\}$ ii. $\{1\}$ iii. $\{-42, 42\}$ iv. $\left\{-\frac{25}{2}, \frac{23}{2}\right\}$

v. $\{3\}$ vi. $\left\{-\frac{78}{5}, \frac{76}{5}\right\}$ vii. $\left\{-\frac{23}{2}, \frac{17}{2}\right\}$ viii. $\{-10, 6\}$

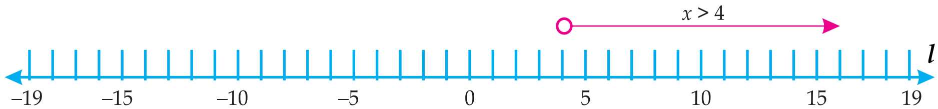
ix. $\left\{\frac{-19}{14}, \frac{-13}{14}\right\}$ x. $\{-41, 44\}$ xi. $\{\ \}$ xii. $\{-4, 3\}$

Exercise 6.3

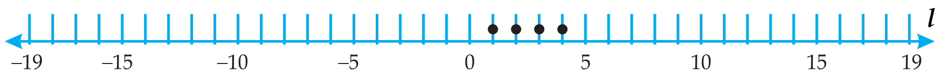
1. i. $\{14, 15, 16, \ldots\}$



ii. $\{x \mid x \in \mathbb{R} \wedge x > 4\}$

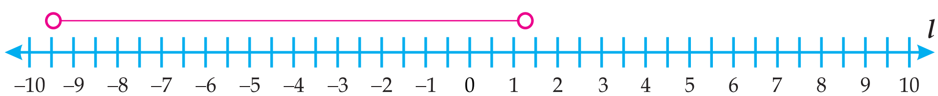


iii. $\{1, 2, 3, 4\}$

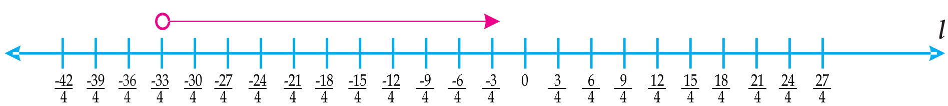


iv. $\{\ \}$

v. $\{y \mid y \in \mathbb{R}\ \frac{-19}{2} < y < \frac{3}{2}\}$



vi. $\{y \mid y \in \mathbb{R} \wedge y > -\frac{33}{4}\}$



2. All the numbers ≥ 4

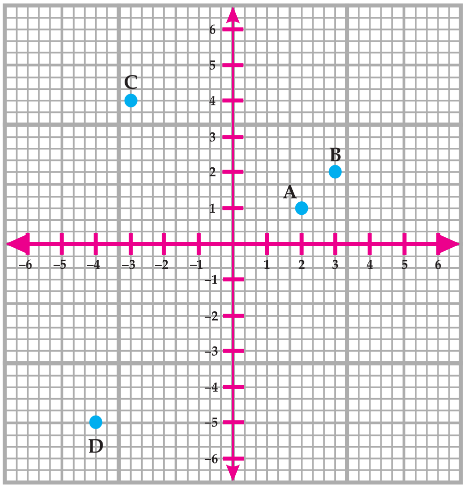
3. Ali must score at least 87 to quality for bonus prize.

Review Exercise 6

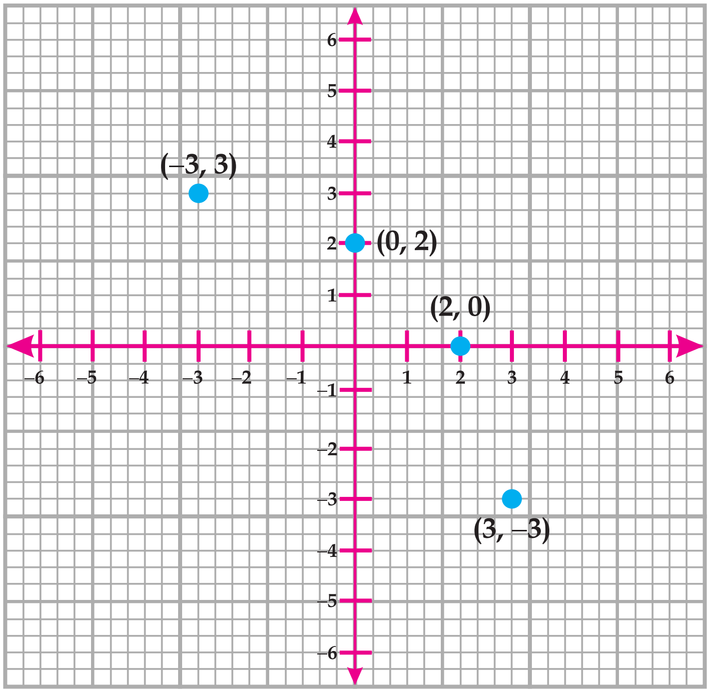
1. i. False ii. False iii. True iv. True v. False
2. i. $\{0\}$ ii. $\{20\}$ iii. $\{\pm 4\}$
   iv. $\{-1\}$ v. $\{y \mid y \in \mathbb{R} \wedge -2 < y < 3\}$
3. i. a ii. a iii. c iv. a v. b vi. c
   vii. c viii. c ix. c x. a

Exercise 7.1

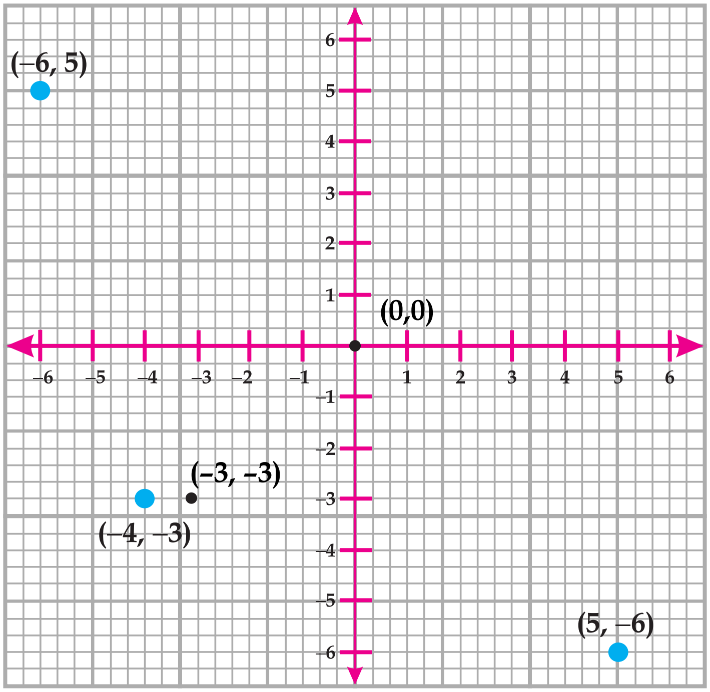
1. i. Abscissa = –2, ordinate = 2 ii. Abscissa = 5, ordinate = –1
   iii. Abscissa = 4, ordinate = 0 iv. Abscissa = –5, ordinate = –6
   v. Abscissa = 3, ordinate = 4 vi. Abscissa = $-\sqrt{8}$, ordinate = $\sqrt{8}$
2. i. Lies in quadrant _ IV ii. Lies in quadrant _ II
   iii. Lies in quadrant - IV iv. Lies in quadrant – III
   v. Lies in quadrant - I vi. Lies in quadrant - I
3. i.



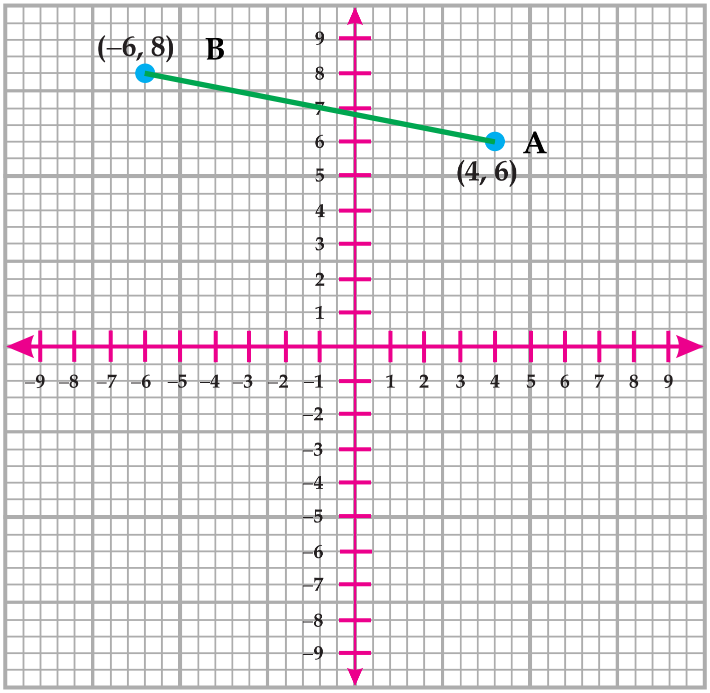
ii.



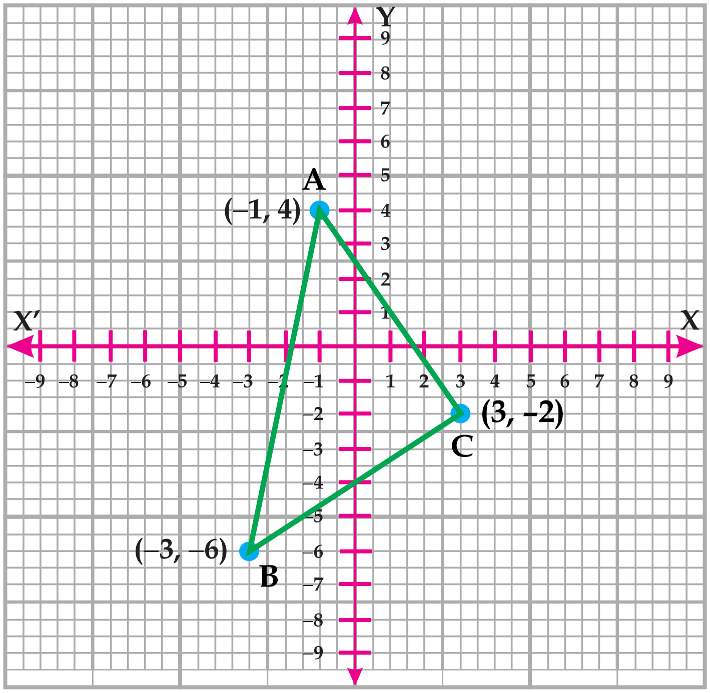
iii.



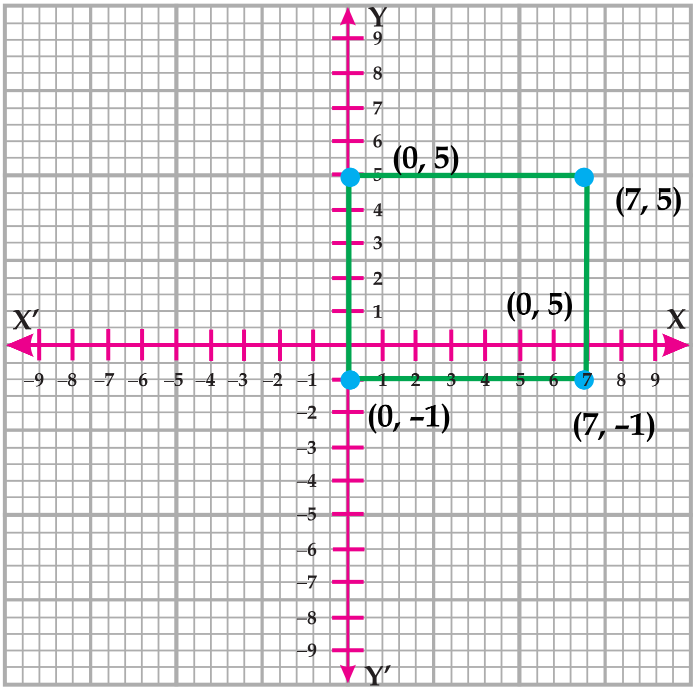
4.



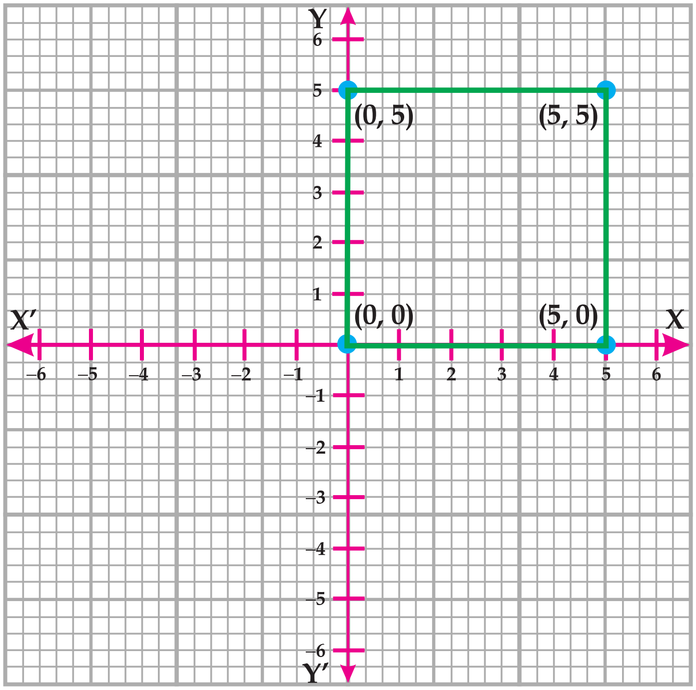
5.



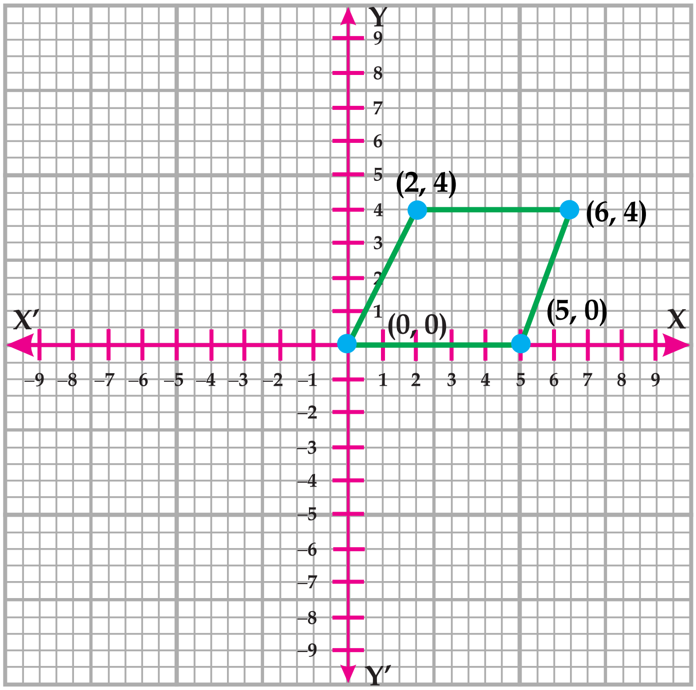
6.



7.



8.



9. **i.** $y = 2 - x$

x	–3	–2	–1	0	1	2	3
y	5	4	3	2	1	0	–1

ii. $y = 2x - 6$

x	–3	–2	–1	0	1	2	3
y	–12	–10	–8	–6	–4	–2	0

iii. $x = 12 - 2y$

y	–3	–2	–1	0	1	2	3
x	18	16	14	12	10	8	6

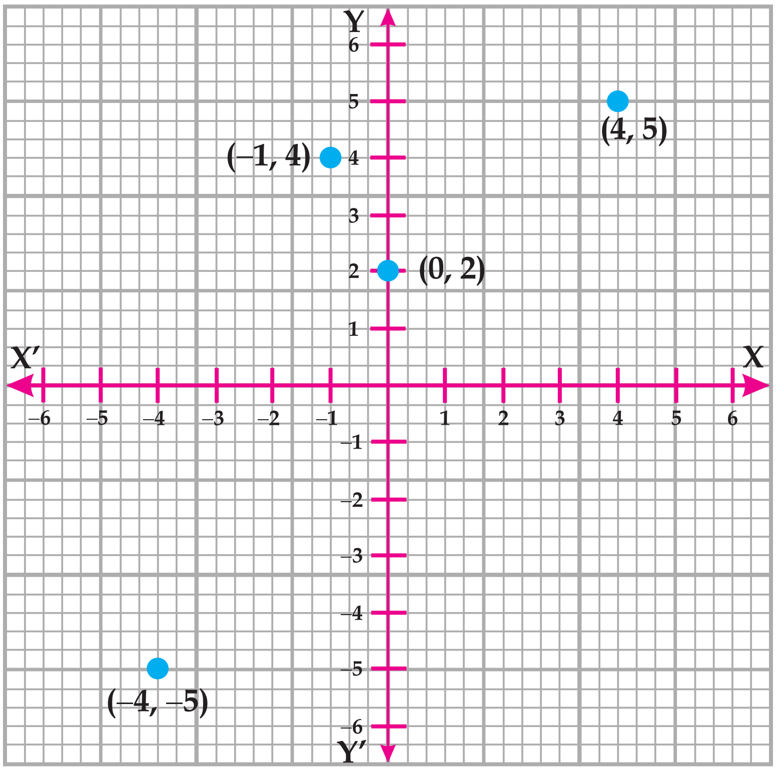
iv. $x = 2y - 3$

y	–3	–2	–1	0	1	2	3
x	–9	–7	–5	–3	–1	1	3

Exercise 7.2

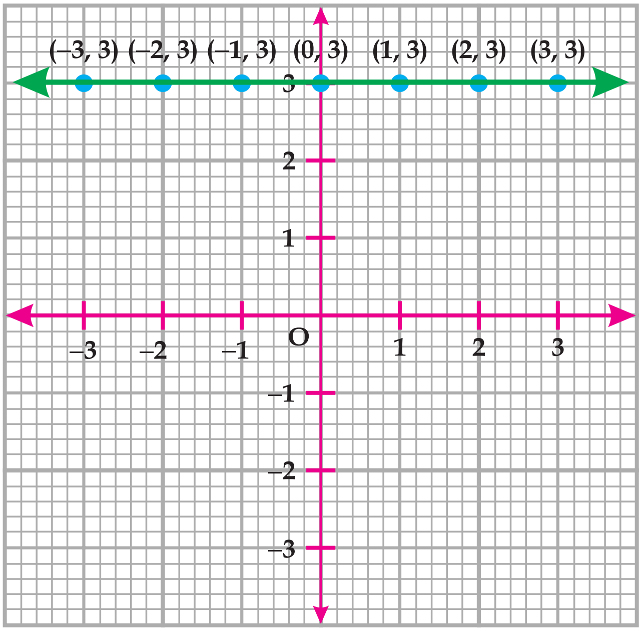
1. $y = 6 - x$

y	–3	–2	–1	0	1	2	3
x	9	8	7	6	5	4	3



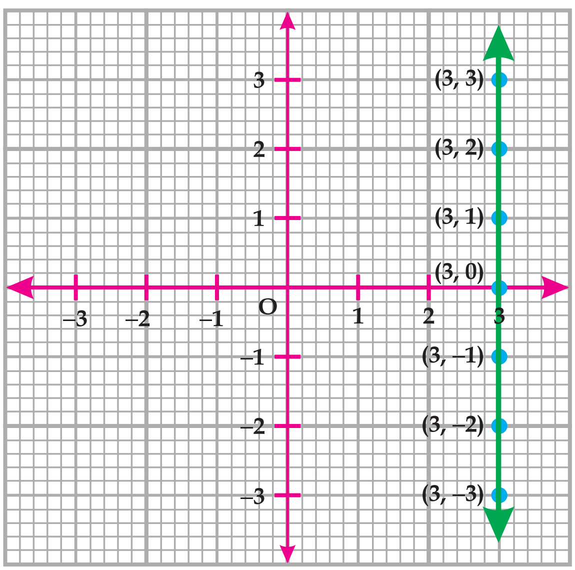
3. i.

x	–3	–2	–1	0	1	2	3
y	3	3	3	3	3	3	3



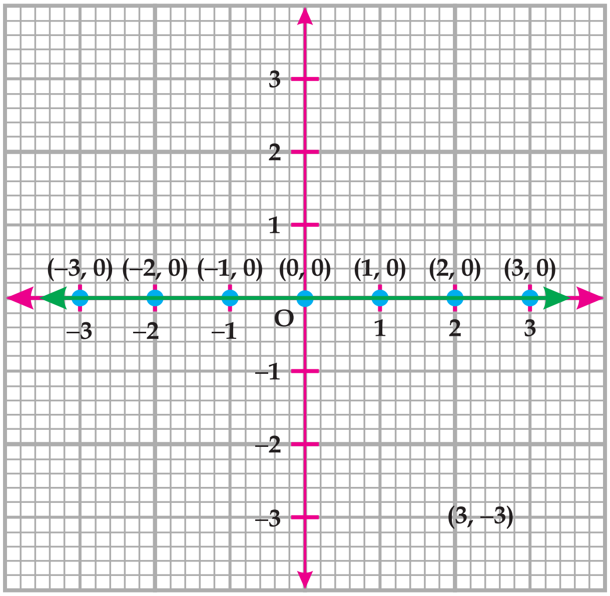
ii.

x	3	3	3	3	3	3	3
y	–3	–2	–1	0	1	2	3



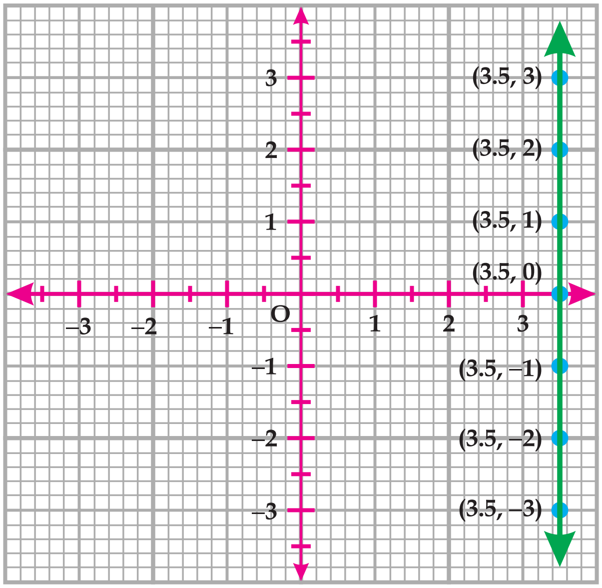
iii.

x	–3	–2	–1	0	1	2	3
y	0	0	0	0	0	0	0



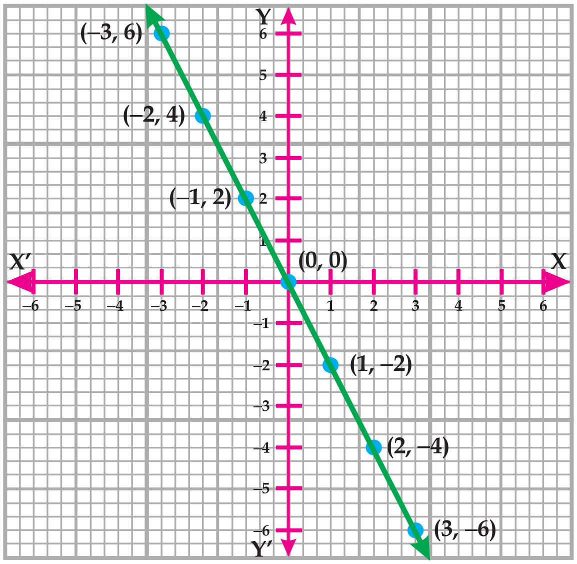
v.

x	3.5	3.5	3.5	3.5	3.5	3.5	3.5
y	–3	–2	–1	0	1	2	3



vi.

x	–3	–2	–1	0	1	2	3
y	6	4	2	0	–2	–4	–6



4. i.

Equation	x-Coordinate	y-Coordinate
$y = \frac{1}{2}x$	0	0
	4	2

ii.

Equation	x -Coordinate	y-Coordinate
$x = \frac{2}{3}y$	1	$\frac{3}{2}$
	1	$\frac{3}{2}$

iii.

Equation	x-Coordinate	y-Coordinate
$2x+4y=8$	0	2
	$\frac{7}{2}$	$\frac{1}{4}$

iv.

Equation	x-Coordinate	y-Coordinate
$2x+y=6$	1	4
	3	0

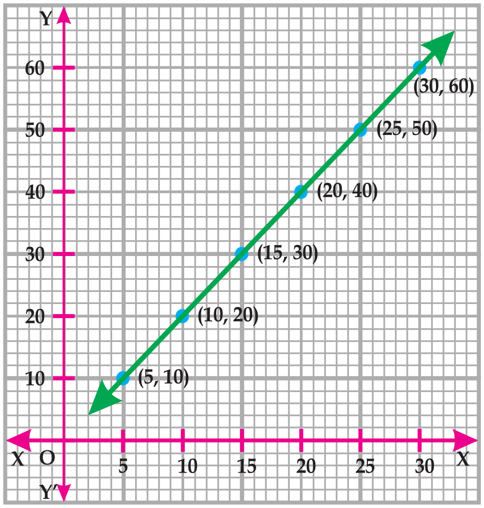
v.

Equation	x-Coordinate	y-Coordinate
$x-y=2$	2	0
	1	–1

vi.

Equation	x -Coordinate	y -Coordinate
$x-3y=6$	3	–1
	3	–1

5.



6. a. The time taken by Ayesha to ride 100 km is 5 hours.
   b. The total distance covered by Ayesha in 3 hours is 60 km.

Exercise 7.3

1. i. 1.6 km ii. 4.8 km iii. 1.2 miles iv. 4.9 miles or 5 miles
2. i. 5 acres ii. 12 acres iii. 2 hectares iv. 6 hectares
3. i. 35.6°F ii. 35.4°F iii. 0°C iv. 2.4°C
4. i. 90° ii. 156 Rs iii. 5 riyal iv. 2.6 riyal
5. i. $\{3,-2\}$ ii. $\{3,1\}$ iii. $\{3,1\}$ iv. $\{4,7\}$ v. $\{1,1\}$
   vi. $\{-3,-4\}$ vii. $\{-3,2\}$ viii. $\{1,1\}$ ix. $\{8,4\}$ x. $\{5,1\}$

Review Exercise 7

1. i. True ii. False iii. False iv. False
3 i. *c* ii. *b* iii. *a* iv. *b* v. *c* vi. *d* vii. *a*

Exercise 8.1

1. i. $\{-2,-3\}$ ii. $\left\{\frac{1}{2},-\frac{1}{3}\right\}$ iii. $\{6,5\}$ iv. $\{2,0\}$

v. $\{5,-3\}$ vi. $\left\{\frac{8}{3},\frac{3}{4}\right\}$ vii. $\{8,2\}$ viii. $\left\{0,-\frac{8}{3}\right\}$

2. i. $\{-3+2\sqrt{2},\ -3-2\sqrt{2}\}$ ii. $\left\{0,-\frac{10}{3}\right\}$ iii. $\left\{\frac{4+\sqrt{13}}{3},\frac{4-\sqrt{13}}{3}\right\}$

iv. $\left\{\frac{3}{4},-\frac{7}{6}\right\}$ v. $\left\{4,-\frac{3}{2}\right\}$ vi. $\left\{\frac{-2+\sqrt{6}}{2},\frac{-2-\sqrt{6}}{2}\right\}$

3. i. $b=-2$ ii. $\frac{-4}{3}$

Exercise 8.2

1. i. $\{5,-3\}$ ii. $\left\{\frac{3}{5},-\frac{5}{2}\right\}$

iii. $\left\{\frac{-1+\sqrt{5}}{2},\frac{-1-\sqrt{5}}{2}\right\}$ iv. $\left\{\frac{-1+2\sqrt{7}}{3},\frac{-1-2\sqrt{7}}{3}\right\}$

v. $\left\{\frac{2+3i\sqrt{5}}{3},\frac{2-3i\sqrt{5}}{3}\right\}$ vi. $\left\{\frac{-3+\sqrt{41}}{4},\frac{-3-\sqrt{41}}{4}\right\}$

vii. $\left\{\frac{1+i\sqrt{5}}{3},\frac{1-i\sqrt{5}}{3}\right\}$ viii. $\left\{\frac{1}{2},\frac{-1}{3}\right\}$

ix. $\left\{\frac{5}{2},0\right\}$ x. $\{1,-1\}$

xi. $\{3\}$ xii. $\{\sqrt{14},-\sqrt{14}\}$

Exercise 8.3

1. $\{\pm 3, \pm i\}$
2. $\{\pm 2, \pm i\}$
3. $\left\{\pm\frac{1}{2}, \pm\frac{\sqrt{6}}{3}\right\}$
4. $\left\{-\frac{3}{4}, -2\right\}$
5. $\left\{\pm\frac{\sqrt{6}\,i}{2}, \pm\frac{2\sqrt{14}\,i}{7}\right\}$
6. $\{-1, 2\}$
7. $\{2\}$
8. $\{-2, 1\}$
9. $\{-1\}$
10. $\{-2\}$
11. $\{2, 4\}$
12. $\left\{3, -10, \frac{-7 \pm \sqrt{79}\,i}{2}\right\}$
13. $\left\{-6, 1, \frac{-5 \pm \sqrt{39}\,i}{2}\right\}$
14. $\left\{0, 1, \frac{1 \pm \sqrt{57}}{2}\right\}$

Exercise 8.4

1. $\{4\}$
2. $\{6\}$
3. $\{1, 3\}$
4. $\{2\}$
5. $\left\{0, \frac{-3}{2}\right\}$
6. $\{0, 3\}$

Review Exercise

1. i. 2 ii. $ax^2 + bx + c = 0,\ a \neq 0$ iii. Exponential
   iv. $x = 2$ v. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2. i. *b* ii. *a* iii. *c* iv. *a* v. *a*
   vi. *c* vii. *b* viii. *a* ix. *c* x. *a*

3. i. False ii. False iii. False iv. False
   v. False vi. True vii. True

Exercise 9.1

3. $x = 17, y = 5$ **4.** $x = 5, y = 3$

Review Exercise 9

1. 30°

2. i. True ii. False iii. True iv. False v. False

3. i. $\overline{DF}$ ii. $\angle RPQ$ iii. Congruent
iv. Equilateral v. Hypotenuse vi. 90°

4. i. *b* ii. *c* iii. *b* iv. *c*

Exercise 10.3

1. 8 cm

Exercise 10.4

3. 1 cm

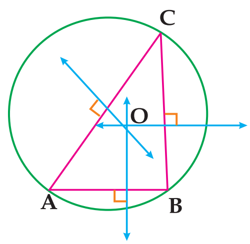
Review Exercise 10

1. i. Congruent ii. Congruent iii. Concurrent iv. Bisect
v. Equal vi. 360°

2. i. *d* ii. *b* iii. *d* iv. *c* v. *b*
vi. *b* vii. *a* viii. *a*

Exercise 11.1

3. The centre of the circle passing through the three non-collinear points is on the point of intersection of right bisector of the line segment obtained by joining these non-collinear points. This point of intersection is equidistant from all three non-collinear points.



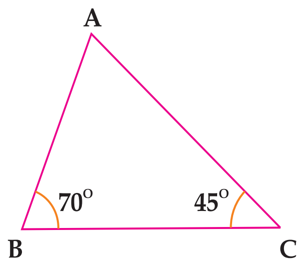
Review Exercise 11

3. i. True ii. True iii. True

4. i. *c* ii. *a* iii. *b* iv. *a*

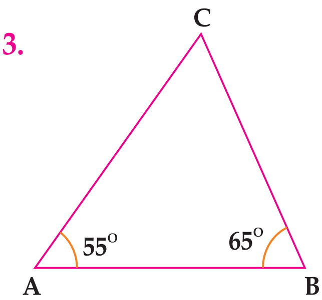
Exercise 12.1

2.



Side AC is longest.

3.



Side BC is smallest.

Review Exercise 12

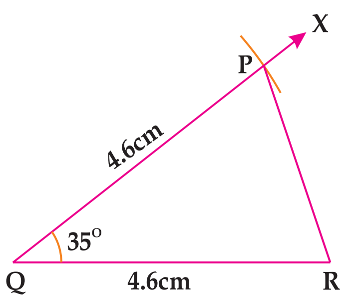
1. i. True ii. False iii. False iv. False v. True

2. i. Hypotenuse ii. Greater iii. $m\overline{AB}$ is smallest

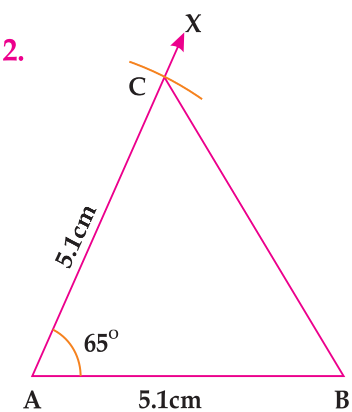
3. i. a ii. c

Exercise 13.1

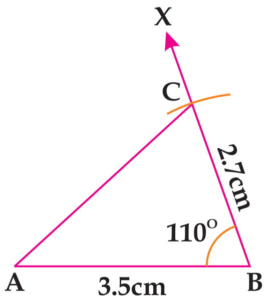
1.



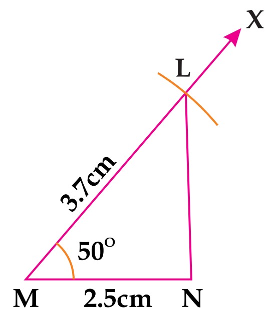
2.



3.

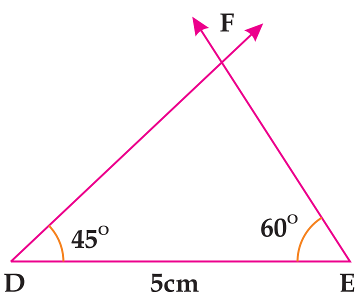

X
C
2.7cm
110°
A
3.5cm
B

4.

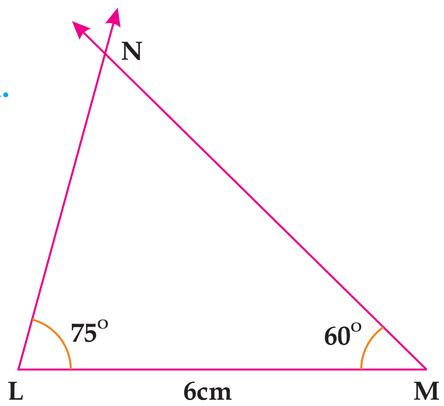

X
L
3.7cm
50°
M
2.5cm
N

6.

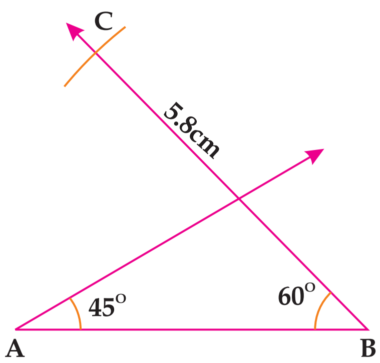
i.


F
45°
60°
D
5cm
E

ii.

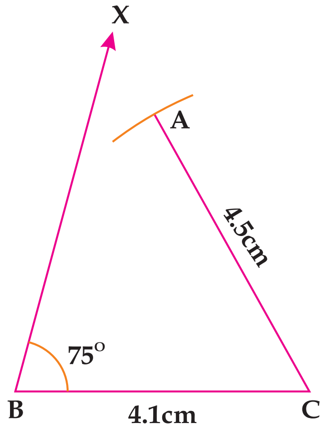

N
75°
60°
L
6cm
M

iii.

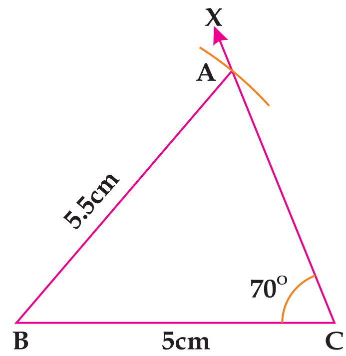

C
5.8cm
45°
60°
A
B

7.

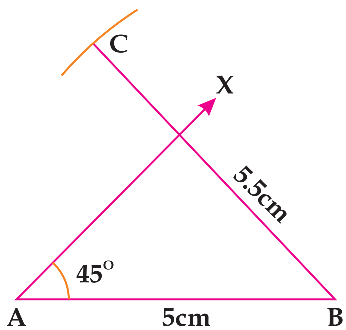
i.



ii.

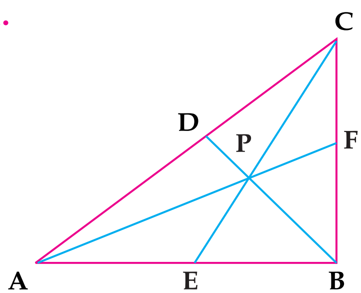


iii.

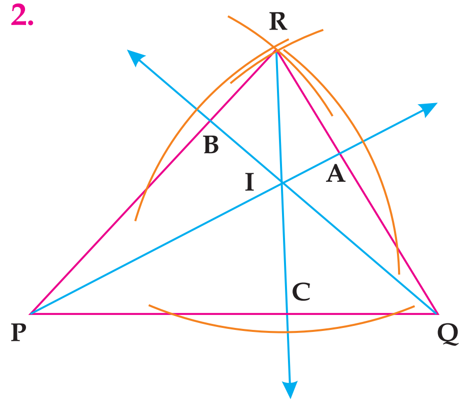


Exercise 13.2

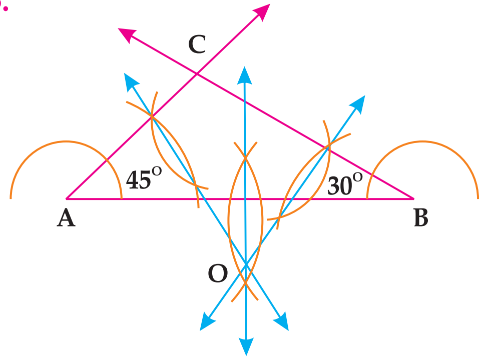
1.



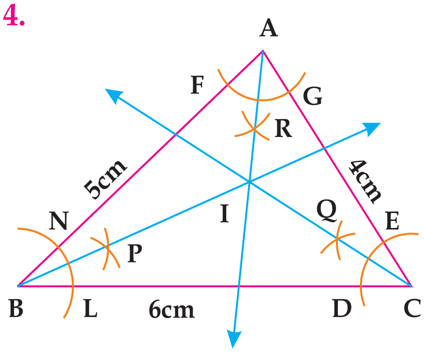
2.



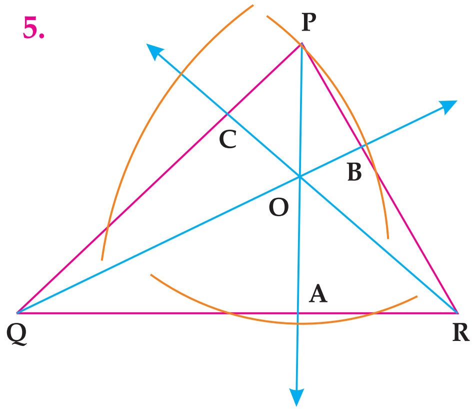
3.


C
45°
30°
A
B
O

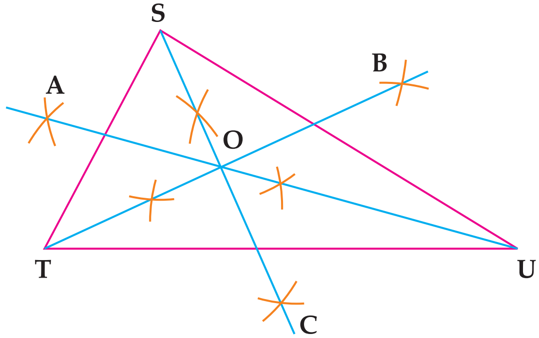
4.


A
F
G
R
5cm
4cm
N
I
Q
E
P
B
L
6cm
D
C

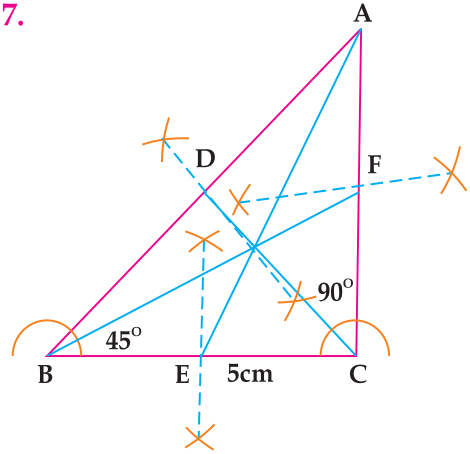
5.


P
C
B
O
A
Q
R

6.

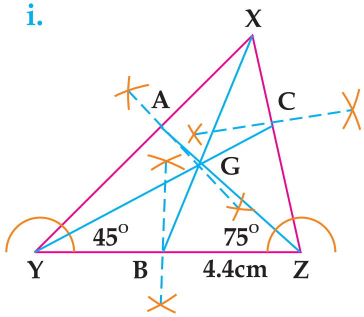

S
B
A
O
T
U
C

7.

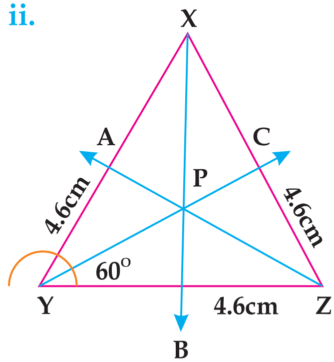

A
D
F
90°
45°
B
E
5cm
C

8.

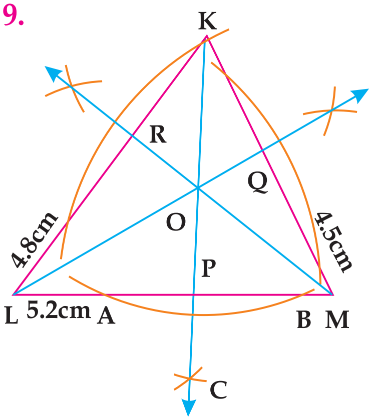
i.



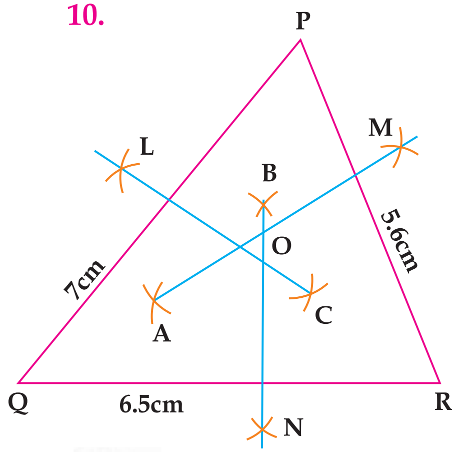
ii.



9.

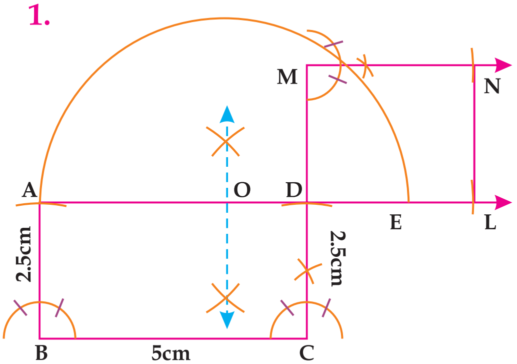


10.

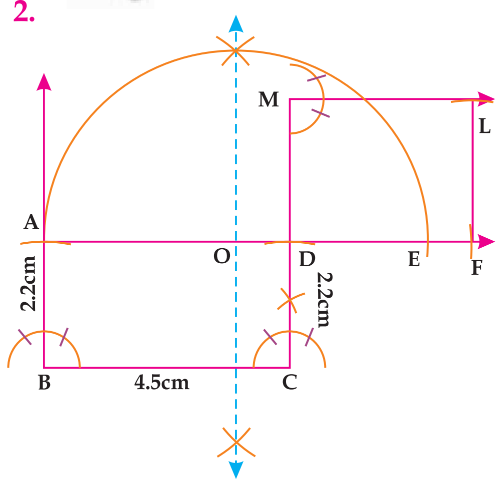


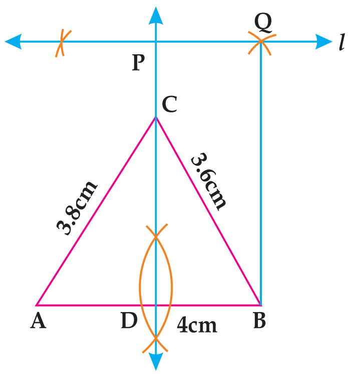
Exercise 13.3

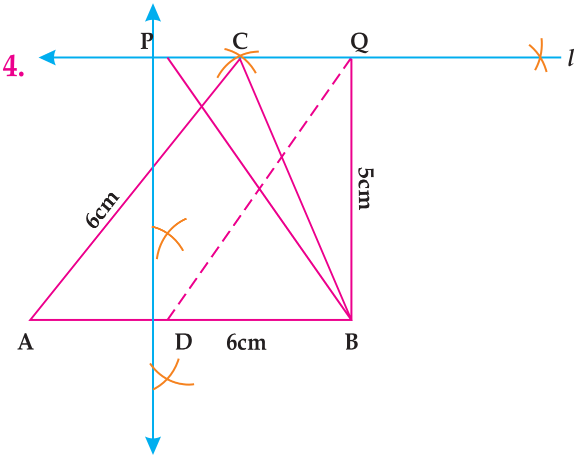
1.



2.



3. 

4. 

Review Exercise 13

1. i. Hypotenuse ii. Altitude of a triangle iii. Median
   iv. Concurrent v. Equal

2. i. *c* ii. *d* iii. *c* iv. *b* v. *a*
   vi. *d* vii. *c* viii. *b* ix. *d*

Exercise 14.1

3. $\blacktriangle BCD = 21\sqrt{2}\ \text{cm}^2$

Review Exercise 14

1. i. True ii. True iii. False iv. True v. False
   vi. False vii. False viii. True

2. i. *b* ii. *b* iii. *c* iv. *d* v. *d* vi. *b*

Exercise 15.1

1. i. Side $m\overline{AB} = 3\sqrt{7}cm$, Area $= \frac{9\sqrt{3}}{2}cm^2$

   ii. Length of side $m\overline{AB} = 4\sqrt{7}cm$, Area $= 8\sqrt{3}cm^2$

2. Length of side $\overline{AC} = \sqrt{116}\,cm$, Area $= 12\,cm^2$
3. Length of side $AC = \sqrt{232}\,cm$, Area $= 24\,cm^2$

Exercise 15.2

2. $m\overline{BC} = 46\ cm$
3. Right triangle
4. length of median $\sqrt{\frac{7}{2}}\,cm$

Review Exercise 15

1. i. $(m\overline{BC})^2$ ii. Isosceles iii. $(m\,\overline{AC})^2$ iv. Triangle
2. $m\overline{AB} = 8cm$

Exercise 16.1

1. i. $\sqrt{101}$ ii. 1 iii. $2\sqrt{2}$ iv. $2\sqrt{2}$
2. i. 5 ii. $\sqrt{117}$ iii. 5 iv. $\sqrt{10}$
3. $P = 12$

Exercise 16.2

4. Points A, B and C forms isosceles triangle.
5. Point A, B and C from right angled triangle.
6. $k = 1 \pm 3\sqrt{3}$
9. Because squares have equal sides and these points determine square.

Exercise 16.3

1. i. $(-1,7)$ ii. $\left(1,\dfrac{1}{2}\right)$ iii. $(-4,3)$ iv. $\left(\sqrt{3},2\sqrt{3}\right)$

2. $(-1,1)$ 3. $B=(2,2)$ 4. center (4, 5)
radius = $5\sqrt{2}m$

Review Exercise 16

1. i. True ii. False iii. False iv. False v. True
vi. True vii. True viii. True ix. False

2. i. $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ ii. Line iii. $y<0$

3. i. c ii. a iii. d iv. d